Some Rational Vehicle Motions

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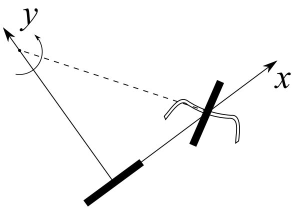
Introduction — Vehicle Motions

New class of special rigid-body motions. Solutions to kinematic equation

$$\frac{dG(t)}{dt} = G(t)S_B$$

Here $G(t) \in SE(3)$ a rigid-body motion and where S_B, the twist velocity in the body-fixed frame is restricted to some fixed screw system.
(Twists - elements of the Lie algebra to $SE(3)$,
screw-system - linear subspace of Lie algebra)

Example — Cars and Bicycles

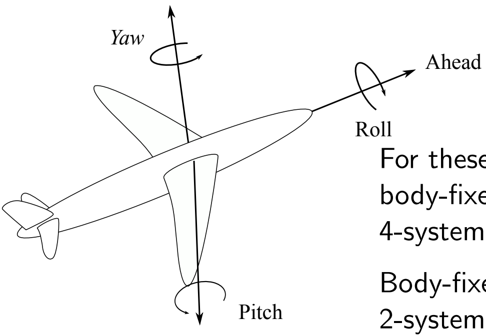


Well studied planar example.

$$S_B = \alpha \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

α forward speed, β turning speed

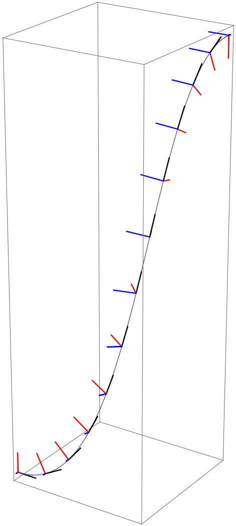
Example — Aeroplanes, Submarines



For these vehicles the possible body-fixed twists are restricted to a 4-system of screws.

Body-fixed twist velocity reciprocal to 2-system of Gibson-Hunt type IIC. Plane can't translate perpendicular to direction of travel.

Example — Frenet-Serret Motion



Not a vehicle motion but same pattern.
Motion determined by the Frenet frame of a curve.

Body-fixed velocity twist lies in a 3-system with Gibson-Hunt type IB_0.

Many others including Needle steering and Bishop Motions

Application I — Simulation

Methods used to study one type of motion can be generalised to other types. Example, substitution made to turn Frenet-Serret equations into Riccarti equations.
Substitution given by Cayley map,

$$G = (I + S)(I - S)^{-1}$$

where $G = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{pmatrix}$ element of Lie algebra with Ω 3×3 anti-symmetric matrix.

Kinematic equation becomes

$$\frac{dS}{dt} = \frac{1}{2}(I + S)S_B(I - S) = \frac{1}{2}(I + SS_B - S_BS - SS_BS)$$

A Riccarti equation.

Application II — Rational Motion

Rational solutions to this Riccarti equation become rational motions. Rearrange,

$$2(I+S)^{-1}\frac{dS}{dt}(I-S)^{-1} = S_B$$

convert to adjoint representation

$$F(\mathbf{s})\frac{d\mathbf{s}}{dt} = \mathbf{s}_B$$

here $\mathbf{s}$ 6×1 vector partitioned as $\mathbf{s}^T = (\boldsymbol{\omega}^T, \mathbf{v}^T)$ with $\boldsymbol{\omega} \times \mathbf{x} = \Omega\mathbf{x}$ for any 3-vector $\mathbf{x}$. Matrix function F is,

$$F(\mathbf{s}) = \frac{2}{1+|\boldsymbol{\omega}|^2}\begin{pmatrix} I-\Omega & 0 \\ V\Omega - V & (1+|\boldsymbol{\omega}|^2)I - \Omega + \Omega^2 \end{pmatrix}$$

Rational Motion, continued

Let $\mathcal{W}_i$ $i = 1, \ldots, n$ be wrenches dual to the screws-system $\mathbf{s}_B$ restricted to. Note $n = 6-$(dimension of screw-system). So

$$\mathcal{W}_i^T F(\mathbf{s}) \frac{d\mathbf{s}}{dt} = 0, \qquad i = 1, \ldots, n$$

Gives n differential equations satisfied by the vehicle motions. Look for rational solutions.

Rational Bicycle Motion

Planar problem, $SE(2)$ not $SE(3)$ hence $n = 3 - 2 = 1$ since $\mathbf{s}_B$ lies in a 2-system. Write

$$\mathbf{s} = \begin{pmatrix} \omega \\ v_x \\ v_y \end{pmatrix}$$

Single kinematic equation is then

$$v_x \dot{\omega} - \omega \dot{v}_x + \dot{v}_y = 0$$

Rational Bicycle Motion — continued

Choosing, $\omega = a_1 t + a_2 t^2$ and $v_x = b_1 t + b_2 t^2$, say, gives

$$v_y = (1/3)(a_1 b_2 - a_2 b_1)t^3$$

(const. of integration 0 so at $t = 0$, $G(0) = I$). Cayley map then gives

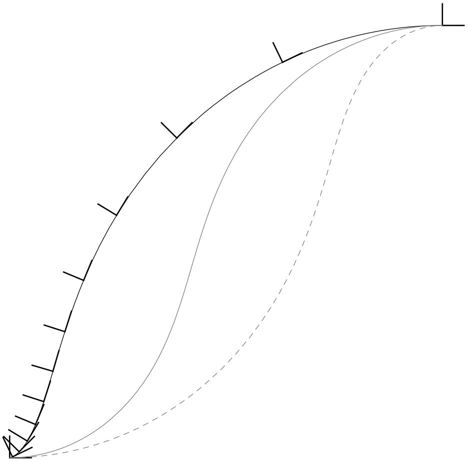
$$G_P(t) = \begin{pmatrix} \frac{(1+a_1t+a_2t^2)(1-a_1t-a_2t^2)}{1+(a_1+a_2t)^2t^2} & \frac{-2(a_1t+a_2t^2)}{1+(a_1+a_2t)^2t^2} & \delta_x \\ \frac{2(a_1t+a_2t^2)}{1+(a_1+a_2t)^2t^2} & \frac{(1+a_1t+a_2t^2)(1-a_1t-a_2t^2)}{1+(a_1+a_2t)^2t^2} & \delta_y \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\delta_x = \frac{2\big(3b_1t + 3b_2t^2 - a_1(a_1b_2 - a_2b_1)t^4 - a_2(a_1b_2 - a_2b_1)t^5\big)}{3\big(1 + (a_1 + a_2t)^2t^2\big)}$$

$$\delta_y = \frac{2\big(3a_1b_1t^2 + (2a_1b_2 + 4a_2b_1)t^3 + 3a_2b_2t^4\big)}{3\big(1 + (a_1 + a_2t)^2t^2\big)}$$

Rational Bicycle Motion — continued



Black curve:

$a_1 = 3$, $a_2 = -3$,
$b_1 = 0$, $b_2 = 1/2$.

Grey curve:

$a_1 = 3$, $a_2 = -3$,
$b_1 = 1$, $b_2 = -1/2$.

Dashed curve:

$a_1 = 3$, $a_2 = -3$,
$b_1 = 2$, $b_2 = -3/2$.

Frenet-Serret Motion

Defined by Frenet frame to a curve. Kinematic equations — Frenet-Serret equations. As above with S_B,

$$S_B = \begin{pmatrix} 0 & -\nu\kappa & 0 & 1 \\ \nu\kappa & 0 & -\nu\tau & 0 \\ 0 & \nu\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where ν, κ and τ are the speed, curvature and torsion of the curve generating the motion.

The adjoint representation of this twist is,

$$\mathbf{s}_B = \alpha \begin{pmatrix} \mathbf{k} \\ \mathbf{0} \end{pmatrix} + \beta \begin{pmatrix} \mathbf{i} \\ \mathbf{0} \end{pmatrix} + \gamma \begin{pmatrix} \mathbf{0} \\ \mathbf{i} \end{pmatrix},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the x, y and z-directions respectively and the coefficients $\alpha = \nu\kappa$, $\beta = \nu\tau$ and $\gamma = \nu$.

Frenet-Serret Motion — Continued

Using Cayley Map to pull-back equations to the Lie algebra get 3 equations:

$$
\begin{aligned}
0 &= \dot{\omega}_y + \omega_x\dot{\omega}_z - \dot{\omega}_x\omega_z, \\
0 &= (\omega_x\omega_y - \omega_z)\dot{v}_x + (1+\omega_y^2)\dot{v}_y + (\omega_x + \omega_y\omega_z)\dot{v}_z + \\
&\qquad (\dot{\omega}_z + \dot{\omega}_x\omega_y - \omega_x\dot{\omega}_y)v_x - (\dot{\omega}_x + \dot{\omega}_y\omega_z - \omega_y\dot{\omega}_z)v_z \\
0 &= (\omega_x\omega_z + \omega_y)\dot{v}_x + (\omega_y\omega_z - \omega_x)\dot{v}_y + (1+\omega_z^2)\dot{v}_z - \\
&\qquad (\dot{\omega}_y + \omega_x\dot{\omega}_z - \dot{\omega}_x\omega_z)v_x + (\dot{\omega}_x + \dot{\omega}_y\omega_z - \omega_y\dot{\omega}_z)v_y.
\end{aligned}
$$

Frenet-Serret Motion — Continued

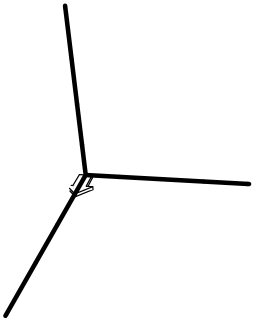
Choosing rational Anzatz can find solutions which give rational motion motion,

$$G_{FS}(t) = \begin{pmatrix} \frac{1+(a^2-b^2)t^2}{1+(a^2+b^2)t^2} & \frac{-2bt}{1+(a^2+b^2)t^2} & \frac{2abt^2}{1+(a^2+b^2)t^2} & \frac{2c}{3}(3t+(a^2-b^2)t^3) \\ \frac{2bt}{1+(a^2+b^2)t^2} & \frac{1-(a^2+b^2)t^2}{1+(a^2+b^2)t^2} & \frac{-2at}{1+(a^2+b^2)t^2} & 2bct^2 \\ \frac{2abt^2}{1+(a^2+b^2)t^2} & \frac{2at}{1+(a^2+b^2)t^2} & \frac{1-(a^2-b^2)t^2}{1+(a^2+b^2)t^2} & \frac{4}{3}abct^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a, b and c arbitrary parameters. Three parameter family of motions.

Frenet-Serret Motion — Continued

Example, $a = b = c = 1/\sqrt{2}$ this example was previously found by Wagner and Ravani.



Conclusions

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THANK YOU