A Characterization of the Dickey-Fuller Distribution With Some Extensions to the Multivariate Case

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Abstract
This paper provides a theoretical functional representation of the density function related to the Dickey-Fuller random variable. The approach is extended to cover the multivariate case in two special frameworks: the independence and the perfect correlation of the series.

Key Words: Dickey-Fuller distribution, unit root

1. Introduction

In this paper we deal with the theoretical case where n possibly cross-dependent time series are generated by

$$\begin{aligned} y_{i,t} &= y_{i,t-1} + u_{i,t} \qquad (i = 1, \ldots, n\,;\ t = 1, \ldots, T) \\ &= y_{i,0} + \sum_{j=1}^{t} u_{i,j} \\ &= y_{i,0} + S_{i,t} \end{aligned} \tag{1}$$

where $y_{i,0} = c_i$ with probability one, or it has a given probability distribution. The $u_{i,t}$'s are assumed to satisfy some regularity conditions so that a suitably normalized transform of $S_{i,t}$, $S^*_{i,T}(r) := T^{-1/2}\sigma_i^{-1}S_{i,\lfloor Tr \rfloor}$ (where $\lfloor \cdot \rfloor$ denotes the integer part and $r \in [0, 1]$), is such that $S^*_{i,T}(r) \Rightarrow W_i(r)$ as $T \to \infty$, with $W_i(r)$ a Wiener process (see e.g. Phillips (1987) for a detailed discussion of such regularity conditions and of the exact meaning of the normalization. Here we take the simplifying assumption that $u_{i,t} \sim \mathrm{iid}(0, \sigma_i^2)$, so that the conditions for the weak convergence of the normalized partial sums are trivially satisfied.
It is well known that, under the null $\mathrm{H}_0 : \rho_i = 1$, the t-ratio based on the OLS estimator of ρ_i, $\widehat{t}_{\rho_i}$, in

$$\Delta y_{i,t} = \rho_i y_{i,t-1} + e_{i,t}\,. \tag{2}$$

has the non-standard Dickey-Fuller limiting distribution

$$\widehat{t}_{\rho_i} \Rightarrow \left(\int_0^1 W_i(r)\mathrm{d}W_i(r)\right)\left(\int_0^1 W_i^2(r)\mathrm{d}r\right)^{-\frac{1}{2}} \tag{3}$$

as $T \to \infty$ (see e.g. Phillips, 1987).
Building on Ruben (1962), in Section 2 we derive an explicit formulation of the conventional univariate Dickey-Fuller distribution (3). Although not easy to manage, the proposed

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formulation can be used to derive analytical results, as opposed to the conventional simulation approach. In this respect, our work is related to Abadir (1995). In Section 3 we extend the analysis to the multivariate case by focusing on the special cases where the series are either independent or perfectly correlated. Section 4 concludes.
In the derivation of the theoretical results, we assume that all the random quantities introduced in the paper are contained in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>0}, P)$.

2. The Univariate Case

Consider the Dickey-Fuller distribution (3): by definition of stochastic integral, we take a partition of the interval $[0,1]$ in N intervals of length $1/N$. If N is large enough, we can approximate the asymptotic distribution of $\widehat{t}_{\rho_i}$ under the null as follows:

$$\left(\int_0^1 W_i(r)\mathrm{d}W_i(r)\right)\left(\int_0^1 W_i^2(r)\mathrm{d}r\right)^{-\frac{1}{2}} \sim$$

$$\sim \left(\sum_{k=1}^{N} W_i(k/N)\left[W_i(k/N) - W_i\left((k-1)/N\right)\right]\right)\left(\frac{1}{N}\sum_{k=1}^{N} W_i^2(k/N)\right)^{-\frac{1}{2}} =$$

$$= \frac{1}{2}\left(W_i^2(1) - 1\right)\left(\frac{1}{N}\sum_{k=1}^{N} W_i^2(k/N)\right)^{-\frac{1}{2}} =: X_i/Y_i, \tag{4}$$

where

$$X_i := \frac{1}{2}\left(W_i^2(1) - 1\right) \tag{5}$$

$$Y_i := \left(\frac{1}{N}\sum_{k=1}^{N} W_i^2(k/N)\right)^{\frac{1}{2}}. \tag{6}$$

Therefore, by definition of Wiener process, the distribution of the univariate Dickey-Fuller test under the stated conditions can be approximated as follows:

$$\widehat{t}_{\rho_i} \Rightarrow X_i/Y_i \sim \frac{\frac{1}{2}\left(\chi^2(1) - 1\right)}{\left(\mathrm{CvM}_0(1)\right)^{\frac{1}{2}}}, \tag{7}$$

where $\chi^2(1)$ denotes a Chi-squared distribution with 1 degree of freedom and $\mathrm{CvM}_0(1)$ denotes a zero level Cramér-von Mises distribution with 1 degree of freedom.
In order to derive an analytical expression for the density function of the Dickey-Fuller distribution, we need to introduce the ratio variable $Z_i := X_i/Y_i$. The cumulative distribution function of Z_i will be denoted as F_i. For $z \in \mathbf{R}$, we have

$$\begin{aligned} F_i(z) &= P(Z_i \le z) \\ &= P(X_i/Y_i \le z) \\ &= \int_0^{+\infty}\int_0^{yz} f_{X_i,Y_i}(x,y)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{8}$$

where f_{X_i,Y_i} is the joint density function of the random variables X_i and Y_i.
The density function f_i of the variable Z_i is

$$\begin{aligned} f_i(z) &= \frac{\partial}{\partial z}\left[\int_0^{+\infty}\int_0^{yz} f_{X_i,Y_i}(x,y)\mathrm{d}x\mathrm{d}y\right] \\ &= \int_0^{+\infty} y f_{X_i,Y_i}(yz,y)\mathrm{d}y. \end{aligned} \tag{9}$$

We need to find an explicit form of the density function f_{X_i,Y_i}, in order to get an expression for f_i. Fixed $x, y \in \mathbf{R}$, we have

$$f_{X_i,Y_i}(x,y) = f_{Y_i|X_i}(y|X_i = x) f_{X_i}(x)\,, \tag{10}$$

where $f_{Y_i|X_i}$ is the density function of Y_i conditional on X_i, and f_{X_i} is the (marginal) density function of the random variable X_i.
We write the cumulative distribution functions of X_i as:

$$\begin{aligned} F_{X_i}(x) &= P(X_i \le x) \\ &= P\left(\frac{1}{2}(W_i^2(1) - 1) \le x\right) \\ &= P(W_i^2(1) \le 2x+1) \\ &= K_{X_i}\int_{-\infty}^{2x+1} s^{-1/2}\mathrm{e}^{-s/2}\mathrm{d}s\,, \end{aligned} \tag{11}$$

where K_{X_i} is the normalizing constant. The density function f_{X_i} is then

$$\begin{aligned} f_{X_i}(x) &= K_{X_i}\frac{\partial}{\partial x}\left[\int_{-\infty}^{2x+1} s^{-1/2}\mathrm{e}^{-s/2}\mathrm{d}s\right] \\ &= 2K_{X_i}(2x+1)^{-1/2}\exp\left[\frac{2x+1}{2}\right]. \end{aligned} \tag{12}$$

The cumulative distribution function of Y_i conditional on $X_i = x$ is

$$\begin{aligned} F_{Y_i|X_i}(y|X_i = x) &= P(Y_i \le y | X_i = x) \\ &= P\left(\left(\frac{1}{N}\sum_{k=1}^{N} W_i^2(k/N)\right)^{1/2} \le y \,\middle|\, W_i^2(1) = 2x+1\right) \\ &= P\left(\left(\frac{1}{N}\left[\sum_{k=1}^{N-1} W_i^2(k/N) + 2x + 1\right]\right)^{1/2} \le y\right) \\ &= P\left(\sum_{k=1}^{N-1} W_i^2(k/N) + 2x + 1 \le Ny^2\right) \\ &= P\left(\sum_{k=1}^{N-1} W_i^2(k/N) \le Ny^2 - 2x - 1\right). \end{aligned} \tag{13}$$

Equation (13) suggests that we need to discuss the distribution of a sum of squared non-independent zero-mean Gaussian variables.
Denote the Cramér-von Mises distribution as

$$V := \sum_{k=1}^{N-1} W_i^2(k/N)\,, \tag{14}$$

and

$$F_V(v) = P(V \le v)\,, \qquad v \in \mathbf{R}^+. \tag{15}$$

Our approach relies on an invariant symmetry property of the random V (see Ruben, 1962). In particular, Ruben (1962) shows that the cumulative distribution function of V can be written as series expansions of Chi-squared cumulative distribution functions, i.e. there exists a sequence of real numbers $\{\lambda_j\}_{j\in\mathbf{N}}$ such that

$$F_V(v) = \sum_{j=0}^{+\infty} \lambda_j F_{N-1+2j}(v)\,, \tag{16}$$

where F_{N-1+2j} is the cumulative distribution function of a Chi-squared random variable with $(N-1+2j)$ degrees of freedom. By substituting the explicit expression of the F's in (16), we obtain

$$F_V(v) = \sum_{j=0}^{+\infty} \lambda_j \int_0^w \mathrm{e}^{-s/2} s^{(N-3+2j)/2} \mathrm{d}s\,, \tag{17}$$

where we assume without loss of generality that the λ's contain also the normalizing constants related to the Chi-squared distributions. By (17), then (13) can be written as

$$\begin{aligned} F_{Y_i|X_i}(y|X_i = x) &= P\left(\sum_{k=1}^{N-1} W_i^2(k/N) \le Ny^2 - 2x - 1\right) \\ &= \sum_{j=0}^{+\infty} \lambda_j \int_0^{Ny^2-2x-1} \mathrm{e}^{-s/2} s^{(N-3+2j)/2} \mathrm{d}s\,. \end{aligned} \tag{18}$$

The conditional density function $f_{Y_i|X_i}$ is

$$\begin{aligned} f_{Y_i|X_i}(y|X_i = x) &= \sum_{j=0}^{+\infty} \lambda_j \frac{\partial}{\partial y}\left[\int_0^{Ny^2-2x-1} \mathrm{e}^{-s/2} s^{(N-3+2j)/2} \mathrm{d}s\right] \\ &= 2Ny \cdot \exp\left[-\frac{Ny^2 - 2x - 1}{2}\right] \times \\ &\quad \times \sum_{j=0}^{+\infty} \lambda_j [Ny^2 - 2x - 1]^{(N-3+2j)/2}\,. \end{aligned} \tag{19}$$

By substituting (12) and (19) into (10), we have

$$\begin{aligned} f_{X_i,Y_i}(x,y) &= 4N\bar{K}_{X_i} \frac{y}{\sqrt{2x+1}} \cdot \exp\left[-\frac{Ny^2 - 2(2x-1)}{2}\right] \times \\ &\quad \times \sum_{j=0}^{+\infty} \lambda_j [Ny^2 - 2x - 1]^{(N-3+2j)/2}\,. \end{aligned} \tag{20}$$

Finally, from (9) and (20) we have

$$\begin{aligned} f_{Z_i}(z) &= 4N\bar{K}_{X_i} \int_0^{+\infty} \frac{y^2}{\sqrt{2yz+1}} \cdot \exp\left[-\frac{Ny^2 - 2(2yz-1)}{2}\right] \times \\ &\quad \times \sum_{j=0}^{+\infty} \lambda_j [Ny^2 - 2yz - 1]^{(N-3+2j)/2} \mathrm{d}y\,. \end{aligned} \tag{21}$$

Although rather involved, (21) can in principle be used to derive analytical results on the Dickey-Fuller distribution.

3. The Multivariate Dickey-Fuller Distribution

This section is devoted to the analysis of the distribution of the multivariate Dickey-Fuller t-ratio.
Let's define the random vector $(Z_1, \ldots, Z_n)$ of asymptotic distributions under the null, accordingly with (3) and (6). More precisely, we can write

$$Z_i := \frac{1}{2}\left(W_i^2(1) - 1\right)\left(\frac{1}{N}\sum_{k=1}^{N} W_i^2(k/N)\right)^{-\frac{1}{2}} \qquad i = 1, \ldots, n\,. \tag{22}$$

Our aim is to provide a closed form expression for the joint density function $f_{Z_1,\ldots,Z_n}$ of the random vector $(Z_1, \ldots, Z_n)$.
Here we study the two extreme cases where the series are either independent or perfectly correlated.

3.1 Independent Series

Assume that the series y's are cross sectional independent. Once that the univariate (marginal) density has been derived, this case becomes trivial. Indeed, for each $(z_1, \ldots, z_n) \in \mathbf{R}^n$, we can write the density function $f_{Z_1,\ldots,Z_n}$ simply as

$$f_{Z_1,\ldots,Z_n}(z_1, \ldots, z_n) = \prod_{i=1}^{n} f_{Z_i}(z_i)\,, \tag{23}$$

where $f_{Z_i}(z_i)$ is given by (21).

3.2 Perfectly Correlated Series

Fixed $i = 2, \ldots, n$, we assume that y_i and y_{i-1} are perfectly correlated, and there exists a constant α_i such that

$$y_i = \alpha_i y_{i-1}\,. \tag{24}$$

Of course, the ordering of the series is purely conventional. Any ordering would be possible just by changing the parameter α_i.
The dependence among the variables is reflected on the dependence among the Wiener processes W_i. In particular, by using the derivation of the Dickey-Fuller asymptotic distribution, then condition (24) can be rewritten in terms of the Wiener processes W_i:

$$W_i = \alpha_i W_{i-1}\,. \tag{25}$$

By substituting (25) into (22), we obtain

$$\begin{aligned} Z_i \;&:=\; \frac{\frac{1}{2}\left((\alpha_i W_{i-1}(1))^2 - 1\right)}{\left(\frac{1}{N}\sum_{k=1}^{N}(\alpha_i^2 W_{i-1}^2(k/N)\right)^{\frac{1}{2}}} \\ &=\; |\alpha_i|\, Z_{i-1} + \frac{\frac{1}{2}(\alpha_i^2 - 1)}{\left(\frac{\alpha_i^2}{N}\sum_{k=1}^{N} W_{i-1}^2(k/N)\right)^{\frac{1}{2}}} \qquad i = 1, \ldots, n\,. \end{aligned} \tag{26}$$

Formula (26) allows us to write explicitly the conditional cumulative distribution of Z_i given Z_{i-1}. Consider $z_i, z_{i-1} \in \mathbf{R}$. Then (18) gives

$$
\begin{aligned}
& P(Z_i \le z_i | Z_{i-1} = z_{i-1}) = \\
& = P\left(|\alpha_i| Z_{i-1} + \frac{\frac{1}{2}(\alpha_i^2 - 1)}{\left(\frac{\alpha_i^2}{N}\sum_{k=1}^N W_{i-1}^2(k/N)\right)^{\frac{1}{2}}} \le z_i \middle| Z_{i-1} = z_{i-1} \right) \\
& = P\left(\sum_{k=1}^N W_{i-1}^2(k/N) \le \left[\frac{2|\alpha_i|(z_i - |\alpha_i| z_{i-1})}{\sqrt{N}(\alpha_i^2 - 1)} \right]^2 \right). \qquad (27)
\end{aligned}
$$

Now, consider the joint density function of the random variable $(Z_1, \dots, Z_n)$. Given $(z_1, \dots, z_n) \in \mathbf{R}^n$, the dependence condition (25) implies

$$
f_{Z_1,\dots,Z_n}(z_1, \dots, z_n) = f_{Z_1}(z_1) \cdot \prod_{i=2}^n f_{Z_i|Z_{i-1}}(z_i | Z_{i-1} = z_{i-1}), \qquad (28)
$$

where

$$
f_{Z_i|Z_{i-1}}(z_i | Z_{i-1} = z_{i-1}) = \frac{\partial}{\partial z_i}\left[P(Z_i \le z_i | Z_{i-1} = z_{i-1}) \right]. \qquad (29)
$$

Consider $i = 1 \dots, n$ and $z_i, z_{i-1} \in \mathbf{R}$. Then (18) gives

$$
P(Z_i \le z_i | Z_{i-1} = z_{i-1}) = \sum_{j=0}^{+\infty} \lambda_j \left[\int_0^{\left[\frac{2|\alpha_i|(z_i - |\alpha_i| z_{i-1})}{\sqrt{N}(\alpha_i^2 - 1)}\right]^2} \mathrm{e}^{-s/2} s^{(N-1+2j)/2} \mathrm{d}s \right]. \qquad (30)
$$

The density function of Z_i conditional on Z_{i-1}, $f_{Z_i|Z_{i-1}}$, is then

$$
\begin{aligned}
f_{Z_i|Z_{i-1}}(z_i | Z_{i-1} = z_{i-1}) &= \sum_{j=0}^{+\infty} \lambda_j \frac{\partial}{\partial z_i} \left[\int_0^{\left[\frac{2|\alpha_i|(z_i - |\alpha_i| z_{i-1})}{\sqrt{N}(\alpha_i^2 - 1)}\right]^2} \mathrm{e}^{-s/2} s^{(N-1+2j)/2} \mathrm{d}s \right] \\
&= \frac{8\alpha_i^2 (z_i - |\alpha_i| z_{i-1})}{N(\alpha_i^2 - 1)^2} \cdot \exp\left\{ -\frac{1}{2} \left[\frac{2|\alpha_i|(z_i - |\alpha_i| z_{i-1})}{\sqrt{N}(\alpha_i^2 - 1)} \right]^2 \right\} \times \\
&\quad \times \sum_{j=0}^{+\infty} \lambda_j \left[\frac{2|\alpha_i|(z_i - |\alpha_i| z_{i-1})}{\sqrt{N}(\alpha_i^2 - 1)} \right]^{N-1+2j}. \qquad (31)
\end{aligned}
$$

4. Conclusions

In this paper an explicit approximation of the density function of the multivariate Dickey-Fuller random variable is provided. We proceed by analyzing at first the univariate case. Our result is grounded on an invariant symmetry property of some random variables involved in the Dickey-Fuller distribution (see Ruben, 1962).

As in Abadir (1995), the followed approach allows us to avoid the conventional simulation-based approach. The theoretical results regarding the univariate case are then extended to the multivariate framework under the assumptions of independent and perfectly correlated series.

Although the independent and the perfectly correlated cases are two extreme settings, they represent the starting point for exploring models with less restrictive assumptions. In this respect, the analysis of a general cross sectional dependence case is already in our research agenda.

Moreover, while we deal here only with the Dickey-Fuller distribution in the absence of deterministic terms, further extensions are under scrutiny to cope with the "constant" and "constant plus linear trend" cases.

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