Equimomental Systems and Robot Dynamics

J.M. Selig

School of Engineering
London South Bank University, U.K.

Introduction

Look at three old problems from a more modern viewpoint.

- How many rigidly connected point masses are needed so that the system has the same inertia properties as an arbitrary rigid body?
- Design a serial robot arm with constant mass matrix.
- Rotor balancing.

Linked by geometry of a Veronese variety, 2-uple embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$.

The Inertia Matrix

Two rigid bodies or systems of point masses are said to be **equimomental** if they have the same inertia matrices (or one can be transformed to the other by a rigid change of coordinates). The 6×6 inertia matrix has the form,

$$N = m \begin{pmatrix} \mathbb{I} & C \\ C^T & I_3 \end{pmatrix}$$

where,

- m is the total mass of the body and I_3 is the 3×3 identity matrix
- C is the position of the body's centre of mass written as an anti-symmetric 3×3 matrix
- $\mathbb{I}$ is the usual 3×3 inertia matrix of the body

4×4 Inertia Matrix

Here more convenient to use a different representation of the inertia,

$$\tilde{\Xi} = m \begin{pmatrix} \frac{1}{2}(-I_{xx} + I_{yy} + I_{zz}) & -I_{xy} & -I_{xz} & c_x \\ -I_{xy} & \frac{1}{2}(I_{xx} - I_{yy} + I_{zz}) & -I_{yz} & c_y \\ -I_{xz} & -I_{yz} & \frac{1}{2}(I_{xx} + I_{yy} - I_{zz}) & c_z \\ c_x & c_y & c_z & 1 \end{pmatrix}$$

where m is the mass of the body as above, c_x and so forth, are the components of the centre of mass and I_{xy} etc. are the components of the 3×3 inertia matrix.

Clearly, two bodies are equimomental if and only if their 4×4 inertia matrices are the same.

Point Masses

The 4×4 inertia matrix of a point with mass m, located at $\mathbf{p} = (p_x,\ p_y,\ p_z)^T$ will be,

$$\tilde{\Xi} = m \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} \begin{pmatrix} p_x & p_y & p_z & 1 \end{pmatrix}$$

Assume the point is in the projective space $\mathbb{P}^3$ with homogeneous coordinates, $\tilde{\mathbf{p}} = (p_x : p_y : p_z : p_0)$, then we can write,

$$\tilde{\Xi} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ p_0 \end{pmatrix} \begin{pmatrix} p_x & p_y & p_z & p_0 \end{pmatrix}$$

(mass not needed here).

The Veronese Variety

Consider the space of all 4×4 symmetric matrices, there is a 10-dimensional vector space of these matrices. Now if we ignore an overall scaling of the matrices the space of these matrices is a 9-dimensional projective space.

The 4×4 inertia matrices form an open set in this $\mathbb{P}^9$. Not all of $\mathbb{P}^9$ since inertia matrices are positive definite.

Point masses lie on the 3-D Veronese variety of rank 1 symmetric matrices.

Four Point Masses

Theorem, probably due to Routh $\sim$ 1870: *there is a system of four point masses of equal mass, equimomental to a general rigid body*.

In terms of the Veronese variety this implies that (almost) any point in $\mathbb{P}^9$ lies on a 3-plane which meets the Veronese variety in 4 points.

Proof—Suppose the mass of the body is m. Take 4 point masses each with mass $m/4$, and place them at the vertices of a regular tetrahedron. The extended position vectors of the points will be,

$$\tilde{\mathbf{p}}_1 = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_2 = \begin{pmatrix} \frac{2\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{-1}{\sqrt{3}} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_3 = \begin{pmatrix} \frac{-\sqrt{2}}{\sqrt{3}} \\ \sqrt{2} \\ \frac{-1}{\sqrt{3}} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_4 = \begin{pmatrix} \frac{-\sqrt{2}}{\sqrt{3}} \\ -\sqrt{2} \\ \frac{-1}{\sqrt{3}} \\ 1 \end{pmatrix}.$$

Proof — continued

Notice that these extended vectors satisfy the relations, $\tilde{\mathbf{p}}_i^T \tilde{\mathbf{p}}_j = 0$ when $i \neq j$, and $\tilde{\mathbf{p}}_i^T \tilde{\mathbf{p}}_i = 4$ for $i = 1, \ldots, 4$. If these 4 points all have masses $m/4$ then the 4×4 inertia matrix of the system will be,

$$\widetilde{\Xi} = \frac{m}{4} \sum_{i=1}^{4} \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_i^T = m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof — concluded

Choose coordinates so that the 6×6 inertia matrix is diagonal, origin at centre of mass, axes aligned with principal axes of inertia. In this coordinate system 4×4 inertia matrix has the form,

$$\widetilde{\Xi} = m \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where a, b and c related to the principal radii of gyration.
The original inertia matrix can be duplicated by subjecting the 4 point-masses to a non-rigid transformation,

$$\tilde{\mathbf{p}}_i' = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{p}}_i, \quad i = 1, 2, 3, 4$$

Remarks

Notice that we could also have subjected the point-masses to an 4-D orthogonal transformation before the non-linear transformation. Suppose $U \in O(4)$ and

$$D = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$\frac{m}{4} \sum_{i=1}^{4} DU\tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_i^T U^T D^T = mDUI_4U^T D^T = mDI_4D^T = \widetilde{\Xi}$$

This implies that the Veronese variety has a 6-parameter family of secant 3-plane through any point in $\mathbb{P}^9$. Since, $\dim\big(O(4)\big) = 6$.

Constant Mass Matrix for Serial Robots

Can summarise equations of motion for a serial robot as,

$$M_{ij}\ddot{\theta}_j + C_{ijk}\dot{\theta}_j\dot{\theta}_k = \tau_i$$

Here summation over repeated indices implied, for 6-joint robot range of sum is $1, \ldots, 6$. θ_i is the ith joint angle and τ_i the torqued applied by the motor at joint i.

M_{ij} is the generalised mass matrix of the robot, when its elements are constant the terms C_{ijk} vanish.

(For simplicity no gravity here).

Elements of the Mass Matrix

The elements of the mass matrix are given by,

$$M_{ij} = \begin{cases} \mathbf{s}_i^T (N_i + \cdots + N_6)\mathbf{s}_j, & \text{if } i \geq j, \\ \mathbf{s}_j^T (N_j + \cdots + N_6)\mathbf{s}_i, & \text{if } i < j, \end{cases}$$

where N_j is the 6×6 inertia matrix of the jth link and $\mathbf{s}_i$ are the twists corresponding to axis of the ith joint,

$$\mathbf{s}_i = \begin{pmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{pmatrix}$$

in patitioned form with $\boldsymbol{\omega}_i$ the angular velocity and $\mathbf{v}_i$ linear velocity.

Can show that the mass matrix will be constant if and only if the composite inertia of all the links above the ith joint are symmetrical about the ith joint.

Symmetry and Balancing

Here “symmetrical about an axis” means “is equimomental to a cylinder about its axis”.

In particular this means that:

- the centre of mass lies on the axis,
- two of the principal moments of inertia are the same and
- the principal axis corresponding to the other principal moment of inertia is the symmetry axis.

Making the mass matrix constant the same problem as balancing a rotor.

Dynamic Balancing

Can use some elementary ideas from Algebraic geometry to show that:
An arbitrary rigid-body can be balanced using two suitably chosen point-masses

To see this note that in $\mathbb{P}^9$ the set of 4×4 inertia matrices which are symmetric with respect to a given axis form a 3-plane, since they are determined by 6 linear equations.

Recall that the point masses form the Veronese variety in $\mathbb{P}^9$, the inertia matrices that can be formed by a pair of point masses lie on lines meeting the Veronese variety in a pair of points. The closure of this space of lines is called the secant variety to the Veronese variety.

Dynamic Balancing — continued

Next given a particular inertia matrix $\widetilde{\widetilde{\Xi}}$, the one to be balanced, we can take the set of 2-planes formed by the secant lines and $\widetilde{\widetilde{\Xi}}$. The variety of all such planes is called the cone over the secant variety with vertex $\widetilde{\widetilde{\Xi}}$.

The dimension of the cone over the secant variety is 7. Using naïve counting arguments the dimension of the secant variety would be 7 but this particular Veronese variety is well known to have a deficient secant variety, in fact the dimension is 6. Taking the cone over the secant variety adds another dimension.

The intersection of the cone over the secant variety with the 3-plane of symmetrical inertias will give points which specify how to balance $\widetilde{\widetilde{\Xi}}$ with 2 point masses. From the above the intersection will have dimension 1 and hence there will be a one-parameter family of solutions.

Conclusions

- Many other results in this area using more (or less) Algebraic geometry.

Conclusions

- Many other results in this area using more (or less) Algebraic geometry.
- Can idea of equimomental systems of points simplify Robot dynamics?

Conclusions

- Many other results in this area using more (or less) Algebraic geometry.
- Can idea of equimomental systems of points simplify Robot dynamics?
- Hope to extend these ideas to synthesising 6×6 stiffness matrices.

Conclusions

- Many other results in this area using more (or less) Algebraic geometry.
- Can idea of equimomental systems of points simplify Robot dynamics?
- Hope to extend these ideas to synthesising 6×6 stiffness matrices.

THANK YOU