A Geometric Newton-Raphson Method for Gough-Stewart Platforms

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Introduction

The forward kinematics of parallel manipulator: Find the rigid-body displacement undergone by the platform given the lengths of the six legs.

Well known to be a hard problem. Much work on this in the past.

Most past work on numerical methods concerns finding all solutions and uses general numerical techniques.

Introduction

Standard numerical methods do not take account of the geometry of the group of rigid-body displacements.

Notice the result we require is a rigid displacement.

Here we present a practical, fast numerical algorithm that finds a single solution given the solution at a nearby position. Method respects the structure of the group of rigid displacements.

Some Notation I

Use 4×4 (homogeneous) representation of the group $SE(3)$.

$$M = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}$$

where R is a 3×3 rotation matrix and $\mathbf{t}$ a translation vector.

Point $\mathbf{p} = (x, y, z)^T$ extended to a 4-D vector $\tilde{\mathbf{p}} = (x, y, z, 1)^T$ so that action on points written,

$$\tilde{\mathbf{p}}' = M\tilde{\mathbf{p}} = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} R\mathbf{p} + \mathbf{t} \\ 1 \end{pmatrix}$$

Notation II

Lie algebra elements can be thought of as 'small' displacements, here errors.
Called twists and given by,

$$S = \left(\frac{d}{dt}M(t)\right) M(t)^{-1} = \begin{pmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{v}$ is the linear velocity of the origin and Ω is a 3×3 anti-symmetric matrix corresponding to the angular velocity of the motion, that is,

$$\Omega\mathbf{p} = \boldsymbol{\omega} \times \mathbf{p}$$

for any $\mathbf{p}$.

Notation III

Twists also written as 6-D vectors,

$$S = \begin{pmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{s} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}$$

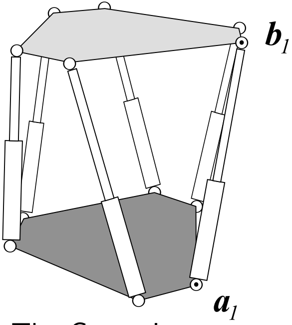
Elements of the dual space to the Lie algebra are called wrenches and written,

$$\mathcal{W} = \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix}$$

where $\mathbf{F}$ is a force and $\boldsymbol{\tau}$ is a moment.

$$\text{power} = \mathcal{W}^T \mathbf{s} = \boldsymbol{\tau} \cdot \boldsymbol{\omega} + \mathbf{F} \cdot \mathbf{v}.$$

The Gough-Stewart Platform



The General Gough-Stewart Platform

The square of the length of the i-leg is given by,

$$l_i^2 = \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right)^T \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right) \qquad i = 1, \ldots, 6$$

Here, $\mathbf{a}_i$ are the centres of the passive joint on the base and $\mathbf{b}_i$ are the centres of the joint on the platform in the home position, that is the position where $M = Id$. The rigid displacement we seek is M here.

Jacobian I

We will need the Jacobian of the manipulator later. To find it we take the derivatives of the leg-lengths,

$$\left.\frac{dl_i^2}{dt}\right|_{t=0} = 2l_i\dot{l}_i = -2\big(\tilde{\mathbf{a}}_i - \tilde{\mathbf{b}}_i\big)^T S\tilde{\mathbf{b}}_i.$$

The matrix S here is the Lie algebra element of the motion, $S = (\dot{M})M^{-1}$. Notice that now we are assuming that $\mathbf{b}_i$ are the point in the current position.

Jacobian II

Rearranging using the cyclic property of the scalar triple product, gives,

$$\dot{l}_i = \frac{1}{l_i}(\tilde{\mathbf{b}}_i - \tilde{\mathbf{a}}_i)^T S\tilde{\mathbf{b}}_i = \frac{1}{l_i}\left((\mathbf{a}_i \times \mathbf{b}_i)^T, (\mathbf{b}_i - \mathbf{a}_i)^T\right)\begin{pmatrix}\omega \\ \mathbf{v}\end{pmatrix}$$

The Jacobian J, is the matrix satisfying,

$$\begin{pmatrix}\dot{l}_1 \\ \vdots \\ \dot{l}_6\end{pmatrix} = J\begin{pmatrix}\omega \\ \mathbf{v}\end{pmatrix}$$

So the rows of this Jacobian are the wrenches,

$$\mathcal{W}_i^T = \frac{1}{l_i}\left((\mathbf{a}_i \times \mathbf{b}_i)^T, (\mathbf{b}_i - \mathbf{a}_i)^T\right), \qquad i = 1, \ldots, 6$$

A Geometric Newton-Raphson Method

Let,

$$L_i = \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right)^T \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right) - l_i^2, \qquad i = 1, \ldots, 6$$

and consider the vector function,

$$\mathbf{F}(M) = \left(L_1, L_2, L_3, L_4, L_5, L_6\right)^T$$

Given the six leg-lengths $l_1, \ldots, l_6$ we seek the rigid transformation M which satisfies $\mathbf{F}(M) = \mathbf{0}$.

The Error Screw

The main idea of this work is to represent the error as a screw. More precisely, if $M^{(i)}$ is the i-th approximation to the solution, then the next approximation will be given by,

$$M^{(i+1)} = e^{S^{(i)}} M^{(i)}$$

where $S^{(i)}$ is the i-th error screw. This recurrence relation forms half of our numerical method. Notice that the result $M^{(i+1)}$ is always a rigid displacement.

Finding the Error Screw I

Consider the Taylor series approximation for the function $\mathbf{F}(e^{tS}M)$ about the root M,

$$\mathbf{F}(e^{tS}M) \approx \mathbf{F}(M) + t\frac{d}{dt}\mathbf{F}(e^{tS}M)_{t=0}$$

Since M is a root of $\mathbf{F}$, $\mathbf{F}(M) = \mathbf{0}$. To compute the derivative of $\mathbf{F}$ we can look at the component functions and as in the previous section,

$$\left.\frac{dL_i}{dt}\right|_{t=0} = -2\big(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\big)^T SM\tilde{\mathbf{b}}_i = 2\big((\mathbf{a}_i \times \mathbf{b}'_i)^T, (\mathbf{b}'_i - \mathbf{a}_i)^T\big)\begin{pmatrix}\boldsymbol{\omega}\\ \mathbf{v}\end{pmatrix}$$

where $\mathbf{b}'_i$ is the position of the point $\mathbf{b}_i$ at the solution.

Finding the Error Screw II

The Taylor expansion can now be written,

$$\mathbf{F}(e^{tS}M) \approx K(M)\mathbf{s}t,$$

where the matrix $K(M) = 2\,\mathrm{diag}(l_1, l_2, \ldots, l_6)J(M)$, with $J(M)$ the Jacobian of the platform.

The error screw $\mathbf{s}$, is found by solving the above equation with $t = 1$, so $\mathbf{s} = -K^{-1}(M)\mathbf{F}(e^S M)$.

As usual with the Newton-Raphson method, we don't know the value of the inverse Jacobian at the solution M so we approximate it by $K^{-1}(M^{(i)})$. This justifies our use of the following recurrence relation for $\mathbf{s}$,

$$\mathbf{s}^{(i)} = -K^{-1}(M^{(i)})\mathbf{F}(M^{(i)})$$

Termination condition

A sensible choice for the condition for iteration to terminate is that the quantity $|\mathbf{F}(M^{(i)})|^2$ be smaller than some predetermined threshold. Notice that this quantity is the sum of the squares of the errors, $L_1^2 + \cdots + L_6^2$.

In practical situations the threshold value should be determined by the accuracy to which the leg-lengths can be measured.

The Algorithm - Inputs

Inputs:

Home position of passive joints $\mathbf{a}_1, \ldots, \mathbf{a}_6$, $\mathbf{b}_1 \ldots, \mathbf{b}_6$,
Current position and orientation $M^{(0)}$,
Desired leg-lengths, $l_1, \ldots, l_6$,
Accuracy threshold, δ.

The Algorithm - Outputs

Outputs:

Position and orientation for desired leg-lengths, M.

The Algorithm - Method

Method:

Compute $\mathbf{F}(M^{(0)})$,

Compute $|\mathbf{F}(M^{(0)})|^2$,

While $\delta > |\mathbf{F}(M^{(i)})|^2$ Repeat:

Evaluate the Jacobian $K(M^{(i)})$,

Compute the error screw,

$$\mathbf{s}^{(i)} = -K^{-1}(M^{(i)})\mathbf{F}(M^{(i)}),$$

Update the position and orientation estimate,

$$M^{(i+1)} = e^{S^{(i)}} M^{(i)},$$

Compute $\mathbf{F}(M^{(i+1)})$,

Compute $|\mathbf{F}(M^{(i+1)})|^2$,

Output $M = M^{(i+1)}$.

Notes on Implementation

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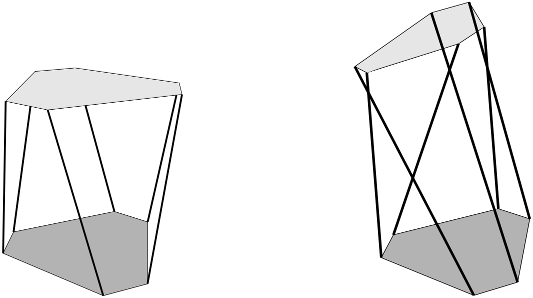
- Quaternions or matrices? Need to multiply group elements, so probably quaternions are simpler.
- The exponential of a screw S can be computed using a degree 3 polynomial in the 4×4 matrix S, similar to the Rodrigues formula for rotations, similar relations can be found for quaternions.

Example



Initial and final pose of the Gough-Stewart Platform for Example

Initial leg-lengths,

$l_1 = 3.1736$, $l_2 = 3.1736$, $l_3 = 3.1736$, $l_4 = 3.1736$, $l_5 = 3.1736$, $l_6 = 3.1736$

Desired final leg-lengths,

$l_1 = 5.7568$, $l_2 = 6.6353$, $l_3 = 7.3836$, $l_4 = 7.1991$, $l_5 = 5.5535$, $l_6 = 6.2567$

Results

Algorithm implemented in *Mathematica*, no attention to efficiency. After 5 iterations, using the identity as the initial value $M^{(0)}$, result is,

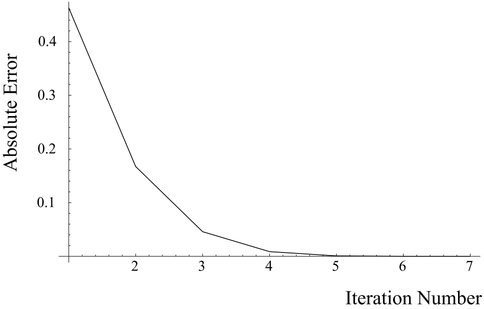
$$M = \begin{pmatrix} 0.4329 & 0.6250 & -0.6495 & -1.0514 \\ -0.7500 & 0.6495 & 0.1250 & 1.6250 \\ 0.5000 & 0.4331 & 0.7500 & 2.7500 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Leg-length errors, (difference between the desired and computed leg-lengths) are,

$$\Delta l_1 = -3.5 \times 10^{-9}, \Delta l_2 = -8.8 \times 10^{-9}, \Delta l_3 = -2.9 \times 10^{-9},$$
$$\Delta l_4 = 5.6 \times 10^{-10}, \Delta l_5 = 1.2 \times 10^{-8}, \Delta l_6 = 1.8 \times 10^{-9}$$

This computation took 0.01s running on a 2GHz Pentium 4 processor with 496MB of RAM.

Another Example



Plot of error in leg 1 against iteration number.

Shows quadratic improvement in error expected of the Newton-Raphson method. Plots of the errors in the other leg-lengths very similar.

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- Could use Cayley map rather than exponential to map errors to the group.
- For some platforms, e.g. 6-3 platform, symbolic inversion of the Jacobian possible.
- Main message — use geometrical numerical methods.