Cayley Maps for SE(3)

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Abstract—*The Cayley map for the rotation group $SO(3)$ is extended to a map from the Lie algebra of the group of rigid body motions $SE(3)$ to the group itself. This is done in several inequivalent ways.*

A close connection between these maps and linear line complexes associated with a finite screw motions is found.

Keywords: Lie algebra, finite screws, line complexes.

I. Introduction

The Cayley map for the rotation group $SO(3)$ is well known, [2]. However, it is not easy to find a brief, clear introduction to the topic. In general, like the exponential map, the Cayley map is a mapping from the Lie algebra of a group to the group itself. However, not all Lie groups have Cayley maps, see for example [4]. The Cayley map for the group of rigid-body motions, $SE(3)$ does not seem to appear in the literature. It will be shown below that the Cayley map for $SE(3)$ is a straightforward generalisation of the original $SO(3)$ Cayley map.

Unlike the exponential map the Cayley map depends on the representation of the Lie algebra that we use. So in fact there are several Cayley maps for $SE(3)$, two cases are studied here, one based on the standard 4×4 representation of $SE(3)$ and the other based on the 6×6 adjoint representation.

The Cayley map is a useful way of linearising the group near its identity. This is also true of the exponential map but the Cayley map is rational, that is it doesn't involve transcendental functions. This is useful in numerical applications since evaluating transcendental functions can be time-consuming. In [3], for example, the Cayley map for $SO(3)$ is considered in connection with the efficient simulation of rigid-body dynamics. Another application of the Cayley map for $SO(3)$ occurs in [6] where interpolated rotational motions are produced in the Lie algebra and then mapped to the group using the Cayley map.

In section IV below an example is given where the Cayley map for $SE(3)$ occurs in a natural geometrical context. It is possible to associate a linear line complex with almost all rigid-body motions. The set of lines in the complex turn out to be reciprocal to the screw or Lie algebra element that produces the rigid-body motions via the Cayley map.

There is another way to associate a linear line complex with a rigid-body motion. This is studied in section VI. It also produces a map from the group to its Lie algebra, and this corresponds to the other Cayley map studied here.

To begin, the Standard Cayley map for $SO(3)$ will be reviewed.

II. The Cayley map for $SO(3)$

A. Definition

The Cayley map is a map from the vector space of 3×3 anti-symmetric matrices to the map of rotation matrices. Let A be a 3×3 anti-symmetric matrix then the Cayley map is defined as:

$$\mathrm{Cay}(A) = (I_3 + A)(I_3 - A)^{-1} = R, \tag{1}$$

where I_3 is the 3×3 identity matrix. It is easy to see that the result of the Cayley map is an orthogonal matrix,

$$\begin{aligned} RR^T &= (I_3 + A)(I_3 - A)^{-1}(I_3 - A)^{-T}(I_3 + A)^T \\ &= (I_3 + A)(I_3 + A)^{-1}(I_3 - A)^{-1}(I_3 - A) \\ &= I_3, \end{aligned} \tag{2}$$

since A is anti-symmetric and $(I_3 + A)$ and $(I_3 - A)$, and their inverses, commute.

Notice that the Cayley map is defined for any 3×3 anti-symmetric matrix. Like the exponential mapping it maps the Lie algebra of the rotation group to the group itself. The group of 3×3 orthogonal matrices $O(3)$ is not connected. It has two connected components, the component containing the identity matrix I_3, corresponds to the group of rotations $SO(3)$. The Cayley map is a continuous map and I_3 is clearly the image of the zero matrix, so we can infer that the image of the Cayley map is always a rotation matrix.

B. Series Expansion

A formula analogous to Rodrigues formula for the exponential map can be derived. All 3×3 anti-symmetric matrices satisfy a cubic equation,

$$A^3 + \lambda^2 A = 0 \tag{3}$$

where $\lambda^2 = -(1/2)\,\mathrm{Tr}(A^2)$. So we can write any power series in A as quadratic polynomial in A. For example, the inverse of $(I_3 - A)$ can be written as,

$$(I_3 - A)^{-1} = I_3 + A + A^2 + A^3 + \cdots, \tag{4}$$

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this converges in some neighbourhood of $A = 0$. Using the cubic equation (3) to simplify higher terms this becomes,

$$(I_3 - A)^{-1} = I_3 + \frac{1}{1+\lambda^2}A + \frac{1}{1+\lambda^2}A^2. \tag{5}$$

Using this result the Cayley map can be written as a quadratic in A,

$$\begin{aligned} \mathrm{Cay}(A) &= (I_3 + A)(I_3 + \frac{1}{1+\lambda^2}A + \frac{1}{1+\lambda^2}A^2) \\ &= I_3 + \frac{2}{1+\lambda^2}A + \frac{2}{1+\lambda^2}A^2. \end{aligned} \tag{6}$$

C. *The Inverse of the Cayley Map*

There are several ways to look at the inverse of the Cayley map. First suppose R is an orthogonal matrix and

$$R = I_3 + \frac{2}{1+\lambda^2}A + \frac{2}{1+\lambda^2}A^2. \tag{7}$$

Since A is anti-symmetric, transposing (7) and taking the result from the original equation gives,

$$R - R^T = \frac{4}{1+\lambda^2}A. \tag{8}$$

Taking the trace of equation (7) gives,

$$\mathrm{Tr}(R) = 3 - \frac{4\lambda^2}{1+\lambda^2}. \tag{9}$$

Eliminating λ from the previous pair of equations gives the result,

$$\mathrm{Cay}^{-1}(R) = A = \frac{1}{1+\mathrm{Tr}(R)}\big(R - R^T\big). \tag{10}$$

Secondly, the original equation defining the Cayley map, (1), could be rearranged. Multiplying by $(I_3 - A)$ gives, $R(I_3 - A) = (I_3 + A)$. Rearranging this gives,

$$\mathrm{Cay}^{-1}(R) = A = (R + I_3)^{-1}(R - I_3). \tag{11}$$

From here the relations

$$(I_3 - A) = 2(R + I_3)^{-1}, \tag{12}$$

and

$$(I_3 + A) = 2(R + I_3)^{-1}R, \tag{13}$$

can also be produced and these will be useful later.

Finally, the 3×3 orthogonal matrices also satisfy a cubic relation, so a quadratic polynomial in R for Cay^{-1} could be derived, however, this will not be pursued here.

Now so long as $1 + \mathrm{Tr}(R) \neq 0$, the inverse of the Cayley map is well defined. If R is a rotation by an angle θ then $\mathrm{Tr}(R) = 1 + 2\cos\theta$. Hence, the map Cay^{-1} is not defined for rotation angles $\pm\pi$. In the region where the map is defined the Cayley map and its inverse are bijective maps, that is the maps are one-to-one and onto. Hence, the 3×3 anti-symmetric matrices can be used as coordinates for the rotations. These coordinate only cover the region where the rotation angle is not $\pm\pi$, so this is just a coordinate patch or chart.

D. *Comparison with the Exponential Map*

For any Lie group the exponential map is a map from the Lie algebra to the group itself. In particular, for $SO(3)$ we have the familiar Rodrigues formula,

$$e^{\theta\Omega} = I + \frac{1}{\theta}\sin\theta\Omega + \frac{1}{\theta^2}(1-\cos\theta)\Omega^2 \tag{14}$$

and its inverse,

$$\ln(R) = \frac{\theta}{2\sin\theta}\big(R - R^T\big) \tag{15}$$

here Ω is a 3×3 anti-symmetric matrix, and $\theta^2 = -(1/2)\,\mathrm{Tr}(\Omega^2)$.

So this map can also be used to define a coordinate patch for the group of rotations. These coordinates are usually known as geodesic coordinates. Notice that, once again, the rotations by $\pm\pi$ lie outside the coordinate patch.

To compare the two maps consider,

$$\mathrm{Cay}^{-1}(e^{\Omega}) = A = \frac{\sin\theta}{\theta(1+\cos\theta)}\Omega \tag{16}$$

Normalising the anti-symmetric matrices gives, $(1/\lambda)A = (1/\theta)\Omega$. For the exponential map this normalised anti-symmetric matrix determines the rotation axis of the corresponding rotation. Hence this will be the same for the Cayley map.

The only difference between the two maps lies in the parameters λ and θ. For the exponential map the parameter θ has physical significance as the angle of rotation. To find the rotation angle in terms of the parameter λ in the Cayley map consider the normalised anti-symmetric matrices and equation (16) above. This gives,

$$\lambda = \frac{\sin\theta}{(1+\cos\theta)} = \tan\frac{\theta}{2}. \tag{17}$$

So if the Cayley map is used then the usual axis/angle representation of the rotations can be recovered simply.

III. A Cayley map for $SE(3)$

In this section the Cayley map will be extended to the group of rigid-body motions $SE(3)$. A typical element of $SE(3)$ can be written as a 4×4 matrix,

$$M = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix},$$

where R is a rotation matrix and $\mathbf{t}$ a translation vector.

In this representation, the Lie algebra elements or twists, can be written,

$$S = \begin{pmatrix} A & \mathbf{u} \\ 0 & 0 \end{pmatrix},$$

where A is an arbitrary 3×3 anti-symmetric matrix and $\mathbf{v}$ is a 3-vector. Now if we simply imitate the $SO(3)$ formula for the Cayley map, (1), we get,

$$(I_4 - S)^{-1}(I_4 + S) = \begin{pmatrix} R & (R + I_3)\mathbf{u} \\ 0 & 1 \end{pmatrix}, \tag{18}$$

where,

$$R = (I_3 - A)^{-1}(I_3 + A)$$

as above. The result (18), can be found by considering the powers of the Lie algebra elements,

$$S^n = \begin{pmatrix} A^n & A^{n-1}\mathbf{u} \\ 0 & 0 \end{pmatrix}.$$

It is clear from (18), that this map is well defined and so we will write,

$$\mathrm{Cay}_4(S) = (I_4 - S)^{-1}(I_4 + S),$$

using the subscript 4 to denote the 4×4 representation of $SE(3)$.

As above a Rodrigues-type formula can be derived for this map. This is based on the fact that the 4×4 matrices S satisfy a degree 4 polynomial equation,

$$S^4 + \lambda^2 S^2 = 0, \tag{19}$$

where, $\lambda^2 = -(1/2)\,\mathrm{Tr}(A^2) = -(1/2)\,\mathrm{Tr}(S^2)$, again. This leads to the formula,

$$\mathrm{Cay}_4(S) = I_4 + 2S + \frac{2}{1+\lambda^2}S^2 + \frac{2}{1+\lambda^2}S^3. \tag{20}$$

The inverse of this Cayley map is easy to compute. Using the partitioned form of M given above gives,

$$\mathrm{Cay}_4^{-1}(M) = \begin{pmatrix} \mathrm{Cay}^{-1}(R) & (R + I_3)^{-1}\mathbf{t} \\ 0 & 0 \end{pmatrix}. \tag{21}$$

As usual this is only defined for rotation angles between $\pm\pi$.

For a general finite screw motion the translational part of the matrix M has the form,

$$\mathbf{t} = (I_3 - R)\mathbf{q} + \left(\frac{p}{2\pi}\right)\boldsymbol{\omega}, \tag{22}$$

where $\mathbf{q}$ is any point on the screw axis, $\boldsymbol{\omega}$ is the direction of the screw axis with $|\boldsymbol{\omega}| = \theta$ the angle of rotation and p is the pitch of the screw. Comparing this with (18) we have,

$$(I_3 + R)\mathbf{u} = (I_3 - R)\mathbf{q} + \left(\frac{p}{2\pi}\right)\boldsymbol{\omega}. \tag{23}$$

Solving for $\mathbf{u}$ gives,

$$\begin{aligned} \mathbf{u} &= (I_3 + R)^{-1}(I_3 - R)\mathbf{q} + \left(\frac{p}{2\pi}\right)(I_3 + R)^{-1}\boldsymbol{\omega} \\ &= -A\mathbf{q} + \left(\frac{p}{4\pi}\right)(I_3 - A)\boldsymbol{\omega} \\ &= \mathbf{q} \times \mathbf{a} + \left(\frac{p}{4\pi}\right)\boldsymbol{\omega} \\ &= \mathbf{q} \times \mathbf{a} + \frac{(\theta/2)}{\tan(\theta/2)}\left(\frac{p}{2\pi}\right)\mathbf{a}, \end{aligned} \tag{24}$$

where the results of section II have been used to simplify these expressions. Notice that the pitch of this screw, $\mathrm{Cay}_4^{-1}(M)$, is the quasi-pitch or 'quatch' referred to in [5].

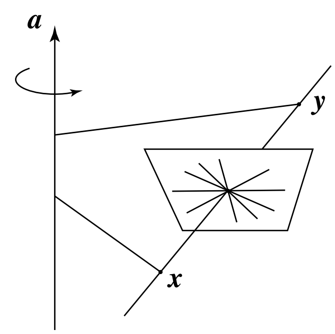

Fig. 1. The Complex of Mid-point Lines.

IV. The Complex of Mid-point lines

Consider a point $\mathbf{x}$ in space. Now apply a rigid transformation to this point, that is an element of $SE(3)$,

$$\mathbf{y} = R\mathbf{x} + \mathbf{t}, \tag{25}$$

where R is a rotation matrix and $\mathbf{t}$ a translation vector. The mid-point of the line joining $\mathbf{x}$ to $\mathbf{y}$ is the point $\mathbf{m} = (1/2)(\mathbf{x} + \mathbf{y})$ and the direction of this line is $\mathbf{v} = \mathbf{y} - \mathbf{x}$. Now shift attention to the lines through the point $\mathbf{m}$ and perpendicular to $\mathbf{v}$, see Fig. 1. It is well known that the set of all such lines, for all possible starting points $\mathbf{x}$, form a linear complex of lines, [1, chap. III]. Here an independent proof of this will be given and the relationship with the Cayley map explained.

Assume that $\boldsymbol{\omega}$ is any vector perpendicular to the direction $\mathbf{v} = \mathbf{y} - \mathbf{x}$, that is $\boldsymbol{\omega} \cdot (\mathbf{y} - \mathbf{x}) = 0$. Then any line through the mid-point $\mathbf{m}$ and perpendicular to the direction $\mathbf{y} - \mathbf{x}$ has Plücker coordinates,

$$\mathbf{s} = \begin{pmatrix} \boldsymbol{\omega} \\ (1/2)(\mathbf{x} + \mathbf{y}) \times \boldsymbol{\omega} \end{pmatrix}, \tag{26}$$

see [7, chap. 6] for an introduction to line geometry in this context. Notice that each point in space is the mid-point of some pair of points $\mathbf{x}$, $\mathbf{y}$. Also, at each point in space we have a point-star of lines through that point. That is, a one dimensional family of lines through the point and all lying in a common plane. This suggests that the set of all these lines lies in a linear complex. To prove this suppose the lines defined above are all reciprocal to a screw $\mathbf{z}$, the equation of the linear complex will be given by,

$$\mathbf{z}^T Q_0 \mathbf{s} = 0, \tag{27}$$

where the 6×6 symmetric matrix is given in partitioned form by,

$$Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}. \tag{28}$$

If the screw $\mathbf{z}$ is given by $\mathbf{z}^T = (\mathbf{a}^T, \mathbf{u}^T)$ then the equation for the line complex becomes,

$$\mathbf{u} \cdot \boldsymbol{\omega} + (1/2)\mathbf{a} \cdot \big((\mathbf{x} + \mathbf{y}) \times \boldsymbol{\omega}\big) = 0. \tag{29}$$

Using the scalar triple product formula this equation can be simplified to,

$$\boldsymbol{\omega}\cdot\Big(\mathbf{u}+(1/2)\mathbf{a}\times(\mathbf{x}+\mathbf{y})\Big)=0. \tag{30}$$

If this is true for all lines perpendicular to $\mathbf{y}-\mathbf{x}$ then we can infer that,

$$\mathbf{u}+(1/2)\mathbf{a}\times(\mathbf{x}+\mathbf{y})=(\gamma/2)(\mathbf{y}-\mathbf{x}), \tag{31}$$

where γ is an arbitrary scalar constant and the factor $1/2$ simplifies some later equations. Next use (25) to write,

$$(1/2)(\mathbf{x}+\mathbf{y})=(1/2)\big((R+I_3)\mathbf{x}+\mathbf{t}\big), \tag{32}$$

and

$$(\mathbf{y}-\mathbf{x})=(R-I_3)\mathbf{x}+\mathbf{t}. \tag{33}$$

Substituting this in (31) gives,

$$\mathbf{u}+(1/2)\mathbf{a}\times\big((R+I_3)\mathbf{x}+\mathbf{t}\big)=(\gamma/2)(R-I_3)\mathbf{x}+\mathbf{t}. \tag{34}$$

This is essentially a system of linear equations which are quite simple to solve symbolically. Let A be the anti-symmetric matrix satisfying, $A\mathbf{p}=\mathbf{a}\times\mathbf{p}$ for any vector $\mathbf{p}$. Then it is simple to see that,

$$A=\gamma(R+I_3)^{-1}(R-I_3). \tag{35}$$

Also we must have,

$$\mathbf{u}=(1/2)\gamma\mathbf{t}-(1/2)\mathbf{a}\times\mathbf{t}. \tag{36}$$

This simplifies to,

$$\mathbf{u}=\gamma(R+I_3)^{-1}\mathbf{t}, \tag{37}$$

since $(I_3-A)=2(R+I_3)^{-1}$, see (12) above.

Notice that this solution depends only on the rotation R and the translation $\mathbf{t}$, not on the point $\mathbf{x}$, hence this set of lines do indeed form a line complex reciprocal to the screw $(\mathbf{a},\mathbf{u})$ found above. The constant γ is irrelevant since the equation for a linear complex is homogeneous. However, it is clear that this screw is the inverse of the Cayley map described in the previous section,

$$\begin{pmatrix}A & \mathbf{u}\\ 0 & 0\end{pmatrix}=\mathrm{Cay}_4^{-1}\begin{pmatrix}R & \mathbf{t}\\ 0 & 1\end{pmatrix}. \tag{38}$$

V. The Cayley Map for the Adjoint Representation

Here the 6×6 representation of the group of rigid-body motions will be used to produce a Cayley map. It will be shown that this map is different from the one found in section III.

A general rigid-body motion will be represented here by the 6×6 matrix,

$$H=\begin{pmatrix}R & 0\\ TR & R\end{pmatrix}, \tag{39}$$

where R is a rotation matrix as usual and the 3×3 matrix T is anti-symmetric and satisfies, $T\mathbf{p}=\mathbf{t}\times\mathbf{p}$ for any vector $\mathbf{p}$. The corresponding representation of the Lie algebra consists of 6×6 matrices of the form,

$$\mathrm{ad}(\mathbf{s})=\begin{pmatrix}A & 0\\ B & A\end{pmatrix}. \tag{40}$$

Both A and B are 3×3 anti-symmetric matrices.

The Cayley map for this representation can be defined as,

$$\mathrm{Cay}(\mathrm{ad}(\mathbf{s}))=(I_6-\mathrm{ad}(\mathbf{s}))^{-1}(I_6+\mathrm{ad}(\mathbf{s})). \tag{41}$$

To be sure that this is well defined we need to show that the result is indeed a rigid transformation as in (39) above. This can be done in a couple of stages, first

$$(I_6-\mathrm{ad}(\mathbf{s}))^{-1}=I_6+\mathrm{ad}(\mathbf{s})+\mathrm{ad}(\mathbf{s})^2+\mathrm{ad}(\mathbf{s})^3+\cdots, \tag{42}$$

in the partitioned form this is,

$$\begin{aligned}(I_6-\mathrm{ad}(\mathbf{s}))^{-1}=&\\ &\begin{pmatrix}I_3 & 0\\ 0 & I_3\end{pmatrix}+\begin{pmatrix}A & 0\\ B & A\end{pmatrix}+\begin{pmatrix}A^2 & 0\\ AB+BA & A^2\end{pmatrix}\\ &+\begin{pmatrix}A^3 & 0\\ A^2B+ABA+BA^2 & A^3\end{pmatrix}+\cdots.\end{aligned} \tag{43}$$

These series can be summed to give,

$$\begin{aligned}(I_6-\mathrm{ad}(\mathbf{s}))^{-1}=&\\ &\begin{pmatrix}(I_3-A)^{-1} & 0\\ (I_3-A)^{-1}B(I_3-A)^{-1} & (I_3-A)^{-1}\end{pmatrix},\end{aligned} \tag{44}$$

then we have,

$$(I_6-\mathrm{ad}(\mathbf{s}))^{-1}(I_6+\mathrm{ad}(\mathbf{s}))=\begin{pmatrix}R & 0\\ X & R\end{pmatrix}, \tag{45}$$

where $R=(I_3-A)^{-1}(I_3+A)$ as in the Cayley map for $SO(3)$ and $X=(I_3-A)^{-1}B(I_3-A)^{-1}(I_3+A)+(I_3-A)^{-1}B$. This last 3×3 matrix can be developed as follows,

$$\begin{aligned}X&=(I_3-A)^{-1}B(I_3-A)^{-1}(I_3+A)+(I_3-A)^{-1}B\\ &=(I_3-A)^{-1}B\big(I_3+(I_3+A)^{-1}(I_3-A)\big)R\\ &=2(I_3-A)^{-1}B(I_3+A)^{-1}R.\end{aligned} \tag{46}$$

Now clearly the matrix $(I_3-A)^{-1}B(I_3+A)^{-1}$ is anti-symmetric, so we may identify,

$$T=2(I_3-A)^{-1}B(I_3+A)^{-1}, \tag{47}$$

and hence conclude that $(I_6-\mathrm{ad}(\mathbf{s}))^{-1}(I_6+\mathrm{ad}(\mathbf{s}))$ is a rigid transformation. So let,

$$\begin{aligned}\mathrm{Cay}_6\left(\mathrm{ad}(\mathbf{s})\right)=&\\ &\begin{pmatrix}\mathrm{Cay}(A) & 0\\ 2(I_3-A)^{-1}B(I_3-A)^{-1} & \mathrm{Cay}(A)\end{pmatrix}.\end{aligned} \tag{48}$$

Again it is possible to write a Rodrigues-type formula for for this map since these 6×6 matrices satisfy the polynomial relation,

$$\mathrm{ad}(\mathbf{s})^5 + 2\lambda^2\,\mathrm{ad}(\mathbf{s})^3 + \lambda^4\,\mathrm{ad}(\mathbf{s}) = 0, \tag{49}$$

where, as before λ is the magnitude of the sub-matrix A; $\lambda^2 = -(1/2)\,\mathrm{Tr}(A^2) = -(1/4)\,\mathrm{Tr}(\mathrm{ad}(\mathbf{s})^2)$. The formula is not elementary this time but can be found using the methods described in [7, §4.4], the result is,

$$\mathrm{Cay}_6(\mathrm{ad}(\mathbf{s})) = I_6 + \frac{2(1+2\lambda^2)}{(1+\lambda^2)^2}\,\mathrm{ad}(\mathbf{s}) + \frac{2(1+2\lambda^2)}{(1+\lambda^2)^2}\,\mathrm{ad}(\mathbf{s})^2 + \frac{2}{(1+\lambda^2)^2}\,\mathrm{ad}(\mathbf{s})^3 + \frac{2}{(1+\lambda^2)^2}\,\mathrm{ad}(\mathbf{s})^4. \tag{50}$$

As above we can rearrange the formulas for the Cayley map to give the inverse Cayley map. As before we have,

$$A = (R + I_3)^{-1}(R - I_3), \tag{51}$$

but now,

$$B = \frac{1}{2}(I_3 - A)T(I_3 + A). \tag{52}$$

Using (12) and (13) above we can write,

$$B = 2(R + I_3)^{-1}TR(R + I_3)^{-1}. \tag{53}$$

Finally here, it will be shown that this map really is different from the one given in section III. To do this consider a particular rigid-body motion but represented in both the 4×4 and 6×6 representations,

$$\begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}.$$

Notice that the rotation matrix R is the same in both cases and the translation vector $\mathbf{t}$ corresponds to the anti-symmetric matrix T, that is $T\mathbf{p} = \mathbf{t} \times \mathbf{p}$ for any vector $\mathbf{p}$. To compare the Lie algebra elements which produce these motions look at the inverse Cayley maps. Clearly the rotational parts of the Lie algebra elements are the same, in both cases we have, $A = (R + I_3)^{-1}(R - I_3)$. For the translational part of the Lie algebra element we have, from (52),

$$B = \frac{1}{2}(T + TA - AT - ATA), \tag{54}$$

for the map Cay_6^{-1}. Using the easily verified relation between vectors and their anti-symmetric matrices; $XY = \mathbf{y}\mathbf{x}^T - (\mathbf{x}\cdot\mathbf{y})I_3$, this relation can be written in terms of vectors,

$$\mathbf{b} = \frac{1}{2}\big((I_3 - A)\mathbf{t} + (\mathbf{a}\cdot\mathbf{t})\mathbf{a}\big) = (R + I_3)^{-1}\mathbf{t} + \frac{1}{2}(\mathbf{a}\cdot\mathbf{t})\mathbf{a}. \tag{55}$$

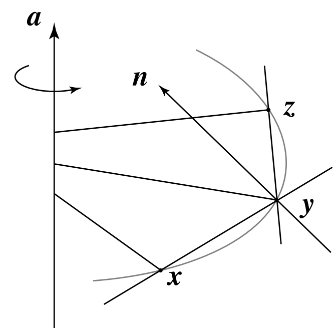

Fig. 2. The Complex of Normal Lines.

The difference between this and $\mathbf{u}$ from the map Cay_4^{-1}, is the term $\frac{1}{2}(\mathbf{a}\cdot\mathbf{t})\mathbf{a}$. This term can be developed using (22),

$$\frac{1}{2}(\mathbf{a}\cdot\mathbf{t})\mathbf{a} = \frac{1}{2}\big(\mathbf{a}^T(I_3 - R)\mathbf{q} + \frac{p}{2\pi}\mathbf{a}\cdot\boldsymbol{\omega}\big)\mathbf{a}. \tag{56}$$

Here, $\mathbf{a}^T(I_3 - R) = \mathbf{0}$ since $\mathbf{a}$ is aligned with the axis of the rotation. So using the relationship between $\mathbf{a}$ and $\boldsymbol{\omega}$ from section II-D, we get,

$$\frac{1}{2}(\mathbf{a}\cdot\mathbf{t})\mathbf{a} = \frac{p}{2\pi}(\theta/2)\tan(\theta/2)\mathbf{a}. \tag{57}$$

Finally this can be combined with the result from (24) and after a little manipulation of the trigonometric expressions we get,

$$\mathbf{b} = \mathbf{q} \times \mathbf{a} + \frac{\theta}{\sin\theta}\left(\frac{p}{2\pi}\right)\mathbf{a}. \tag{58}$$

That is $\mathrm{Cay}_6^{-1}(\mathrm{ad}(\mathbf{s}))$ has the same axis as the corresponding $\mathrm{Cay}_4^{-1}(M)$ but the pitches are different.

VI. Another Line Complex

In [1, Chap. III] another line complex associated to a finite screw motion is described. There are several geometrical descriptions of this complex, here a description based on points will be given.

Let $\mathbf{x}$ be an arbitrary point in space and let $\mathbf{y}$ be the image of $\mathbf{x}$ under the rigid motion, so, $\mathbf{y} = R\mathbf{x} + \mathbf{t}$ as before. Now consider also the image of $\mathbf{y}$ under the same rigid motion, call it $\mathbf{z}$ so that $\mathbf{z} = R\mathbf{y} + \mathbf{t}$. These three points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ determine a plane and the lines of the complex under discussion are normal to this plane through the second point $\mathbf{y}$, see Fig. 2. The direction of such a line is given by the vector product,

$$\mathbf{n} = (\mathbf{z} - \mathbf{y}) \times (\mathbf{y} - \mathbf{x}) = \mathbf{z}\times\mathbf{y} + \mathbf{y}\times\mathbf{x} + \mathbf{x}\times\mathbf{z}. \tag{59}$$

The moment of the line is simply $\mathbf{y} \times \mathbf{n}$. Next assume that all such lines are reciprocal to a screw $\mathbf{s}^T = (\mathbf{c}^T, \mathbf{w}^T)$. That is,

$$\mathbf{c}\cdot(\mathbf{y}\times\mathbf{n}) + \mathbf{w}\cdot\mathbf{n} = 0, \tag{60}$$

again the scalar triple product formula can be used to simplify this,

$$(\mathbf{c}\times\mathbf{y} + \mathbf{w})\cdot\mathbf{n} = 0. \tag{61}$$

This implies that $(\mathbf{c}\times\mathbf{y}+\mathbf{w})$ is normal to $\mathbf{n}$ and hence it can be written as a linear sum in the pair of vectors used to produce $\mathbf{n}$,

$$\mathbf{c}\times\mathbf{y}+\mathbf{w}=\alpha(\mathbf{z}-\mathbf{y})+\beta(\mathbf{y}-\mathbf{x}), \qquad (62)$$

where α and β are constants to be determined. Now the points $\mathbf{y}$ and $\mathbf{z}$ can be rewritten in terms of $\mathbf{x}$,

$$\mathbf{c}\times R\mathbf{x}+\mathbf{c}\times\mathbf{t}+\mathbf{w}=(R-I_3)(\alpha R+\beta I_3)\mathbf{x}+(\alpha R+\beta I_3)\mathbf{t}. \qquad (63)$$

Since $\mathbf{x}$ is arbitrary we must have,

$$CR=(R-I_3)(\alpha R+\beta I_3), \qquad (64)$$

and multiplying on the right by the transpose of the rotation matrix gives,

$$C=\big(\alpha R+(\beta-\alpha)I_3-\beta R^T\big). \qquad (65)$$

For C to be anti-symmetric, as required, clearly means that $\alpha=\beta$. So $C=\alpha(R-R^T)$ and we see that the axis of the screw that defines the complex is again in the direction of the axis of the finite screw motion. The translational part of the screw is then given by,

$$\mathbf{w}=\alpha(R+I_3)\mathbf{t}-C\mathbf{t}=\alpha(I_3+R^T)\mathbf{t}. \qquad (66)$$

Substituting for $\mathbf{t}$ using equation (22) gives,

$$\mathbf{w}=\Big(\mathbf{q}\times\mathbf{c}+\frac{\theta}{\sin(\theta)}\Big(\frac{p}{2\pi}\Big)\mathbf{c}\Big). \qquad (67)$$

This the same as the screw defined by Cay_6^{-1}, it has the same axis and pitch, see (58). The vector $\mathbf{c}$ is only defined up to an overall constant α, but this is immaterial since the equation for the complex is homogeneous.

VII. Conclusions

The Lie algebra of the group of rigid body motions consists of infinitesimal screws or twists. In this work two maps have been found from this Lie algebra to the group of rigid-body motions. These maps are different generalisations of the Cayley map on $SO(3)$ and appear to be novel. Different representations of $SE(3)$ will presumably produce more such maps.

However, as shown above, there is a natural connection between these Cayley maps and linear line complexes. Bottema and Roth study two such complexes associated with a finite screw motion. They call these two complexes Γ and Γ' [1]. The complex Γ is the complex of mid-point lines associated with the Cayley map Cay_4. While Γ', the complex of normal lines is associated with Cay_6. This connection is also believed to be novel. Both of these Cayley maps associate a finite screw motion with an infinitesimal screw with the same axis, the difference between the maps is what happens to the pitches of the screws. Only the exponential map sends a pitch p infinitesimal screw to finite screw with the same pitch.

Cayley maps can often be used as the basis for efficient numerical methods in place of the exponential map. Note that the Lie algebra structure on the infinitesimal screws is derived from the group structure on the finite screws using the exponential map. Hence, it may be possible to produce a deformed Lie algebra using a Cayley map. This might be useful for speeding up the numerical computations used in robot control.

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Note: This is a corrected version of a paper presented at the 12th IFToMM World Congress, Besançon, June 2007. An error in the original gave the wrong expression for the pitch of the second line complex and hence prevented the identification between this line complex and the Cayley map derived from the adjoint representation of the group.