Cayley Maps for $SE(3)$

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Introduction

Cayley map well known for rotations, $SO(3)$.

Want to extend this to the group of rigid body motions $SE(3)$.

Close connection with linear line complexes.

The Cayley map for $SO(3)$

Let A be a 3×3 anti-symmetric matrix, then the Cayley map is defined as,

$$\mathrm{Cay}(A) = (I_3 + A)(I_3 - A)^{-1} = R$$

It is a straightforward exercise to show that R is a rotation matrix.

The inverse Cayley map for $SO(3)$

Not every rotation occurs in this way, the Cayley map is not onto.
Rearranging the previous equation gives,

$$A = (R + I_3)^{-1}(R - I_3) = \mathrm{Cay}^{-1}(R)$$

Can't be solved if $(R + I_3)$ is singular, that is if rotation angle is $\pm\pi$.

Cayley map very similar to exponential map—a map from the Lie algebra of the group to the group itself.

The Exponential map

For any 3×3 antisymmetric matrix Ω we have,

$$e^{\Omega} = I_3 + \Omega + \frac{\Omega^2}{2!} + \frac{\Omega^3}{3!} + \cdots + \frac{\Omega^n}{n!} + \cdots$$

which can be abbreviated to the Rodrigues formula using the cubic relation satisfied by the anti-symmetric matrices,

$$e^{\Omega} = I_3 + \frac{1}{\theta}\sin\theta\Omega + \frac{1}{\theta^2}(1 - \cos\theta)\Omega^2$$

where $\mathrm{Tr}(\Omega^2) = -2\theta^2$.

Formula for the Cayley map

Easy to produce Rodrigues like formulas for the Cayley map, since anti-symmetric matrix A also satisfies cubic relation.

$$\mathrm{Cay}(A) = I_3 + \frac{2}{1+\lambda^2}A + \frac{2}{1+\lambda^2}A^2$$

where $\mathrm{Tr}(A^2) = -2\lambda^2$.

Also the inverse Cayley map can be written,

$$\mathrm{Cay}^{-1}(R) = A = \frac{1}{1+\mathrm{Tr}(R)}(R - R^T)$$

Comparison with the Exponential

Now we can compare the Cayley map with the exponential map,

$$\mathrm{Cay}^{-1}(e^{\Omega}) = A = \frac{\sin\theta}{\theta(1+\cos\theta)}\Omega$$

so A determines rotation axis (like Ω).
For exponential, θ is rotation angle. Comparing normalised matrices $(1/\lambda)A = (1/\theta)\Omega$,

$$\lambda = \frac{\sin\theta}{(1+\cos\theta)} = \tan\frac{\theta}{2}$$

A Cayley map for $SE(3)$

Use homogeneous rep. of $SE(3)$, group elements are 4×4 matrices,

$$M = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix},$$

where R a rotation matrix and $\mathbf{t}$ a translation vector. Corresponding Lie algebra elements or twists (screws),

$$S = \begin{pmatrix} A & \mathbf{u} \\ 0 & 0 \end{pmatrix},$$

A — 3×3, anti-symmetric, $\mathbf{u}$ — 3-vector.

A Cayley map for $SE(3)$ (cont.)

Simply extend definition above,

$$\mathrm{Cay}_4(S) = (I_4 - S)^{-1}(I_4 + S),$$

can show this is well defined etc.
Rodriguies formula,

$$\mathrm{Cay}_4(S) = I_4 + 2S + \frac{2}{1+\lambda^2}S^2 + \frac{2}{1+\lambda^2}S^3$$

where, $\lambda^2 = -(1/2)\,\mathrm{Tr}(A^2) = -(1/2)\,\mathrm{Tr}(S^2)$.

Inverse of the Cayley map

Inverse of this Cayley map easy to compute.

$$\mathrm{Cay}_4^{-1}(M) = \begin{pmatrix} \mathrm{Cay}^{-1}(R) & (R + I_3)^{-1}\mathbf{t} \\ 0 & 0 \end{pmatrix}$$

for rotation angles between $\pm\pi$ only.
Translational part of general finite screw,

$$\mathbf{t} = (I_3 - R)\mathbf{q} + \left(\frac{p}{2\pi}\right)\boldsymbol{\omega}$$

$\mathbf{q}$ any point on the screw axis, $\boldsymbol{\omega}$ the direction of the screw axis, $|\boldsymbol{\omega}| = \theta$ rotation angle, p the pitch of the screw.

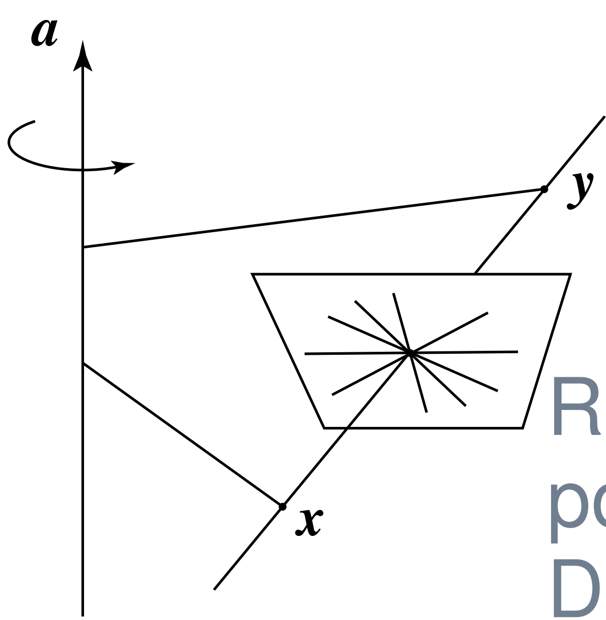
Pitch of the screw

Translational part of the infinitesimal screw is,

$$\begin{aligned}
\mathbf{u} &= (I_3+R)^{-1}(I_3-R)\mathbf{q} + \left(\frac{p}{2\pi}\right)(I_3+R)^{-1}\boldsymbol{\omega} \\
&= -A\mathbf{q} + \left(\frac{p}{4\pi}\right)(I_3-A)\boldsymbol{\omega} \\
&= \mathbf{q}\times\mathbf{a} + \left(\frac{p}{4\pi}\right)\boldsymbol{\omega} \\
&= \mathbf{q}\times\mathbf{a} + \frac{(\theta/2)}{\tan(\theta/2)}\left(\frac{p}{2\pi}\right)\mathbf{a}
\end{aligned}$$

$\mathbf{a}$—3-vector corresponding to A.
Pitch of this infinitesimal screw is quasi-pitch or 'quatch' introduced by Hunt and Parkin (1995).

The Complex of Midpoint lines



Rotate and translate point $\mathbf{x}$ to point $\mathbf{y} = R\mathbf{x} + \mathbf{t}$.
Direction of the line joining these points is $\mathbf{v} = \mathbf{y} - \mathbf{x}$ and the midpoint between them is $\mathbf{m} = (1/2)(\mathbf{x} + \mathbf{y})$.

Plücker Coordinates

Set of lines through $\mathbf{m}$ and perpendicular to $\mathbf{v}$ for all points $\mathbf{x}$ form a linear line complex—Bottema and Roth.
The Plücker coordinates of such lines are,

$$\mathbf{s} = \begin{pmatrix} \boldsymbol{\omega} \\ (1/2)(\mathbf{x}+\mathbf{y}) \times \boldsymbol{\omega} \end{pmatrix},$$

where $\boldsymbol{\omega} \cdot \mathbf{v} = 0$.

Linear Line complexes

If the lines s lie in a linear line complex they will all be reciprocal to some screw $\mathbf{z} = \begin{pmatrix} \mathbf{a} \\ \mathbf{u} \end{pmatrix}$,

$$\mathbf{z}^T Q_0 \mathbf{s} = 0$$

where $Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$, the Klein form on the Lie algebra.

Substituting into the above equation gives,

$$\mathbf{u}\cdot\boldsymbol{\omega}+(1/2)\mathbf{a}\cdot((\mathbf{x}+\mathbf{y})\times\boldsymbol{\omega})=0$$

Use scalar triple product formula to rearrange,

$$\boldsymbol{\omega}\cdot(\mathbf{u}+(1/2)\mathbf{a}\times(\mathbf{x}+\mathbf{y}))=0$$

If true for all lines perpendicular to $\mathbf{v}=\mathbf{y}-\mathbf{x}$ then,

$$\mathbf{u}+(1/2)\mathbf{a}\times(\mathbf{x}+\mathbf{y})=(\gamma/2)(\mathbf{y}-\mathbf{x})$$

γ arbitrary

Now substitute for $\mathbf{y} = R\mathbf{x} + \mathbf{t}$,

$$\mathbf{m} = (1/2)(\mathbf{x} + \mathbf{y}) = (1/2)((R + I_3)\mathbf{x} + \mathbf{t})$$

and

$$\mathbf{v} = (\mathbf{y} - \mathbf{x}) = (R - I_3)\mathbf{x} + \mathbf{t}$$

Equation becomes,

$$\mathbf{u} + (1/2)\mathbf{a} \times ((R + I_3)\mathbf{x} + \mathbf{t}) = (\gamma/2)(R - I_3)\mathbf{x} + \mathbf{t}.$$

Cayley map again

Simple to solve for $\mathbf{a}$ and $\mathbf{u}$ if we write A for the 3×3 anti-symmetric matrix corresponding to $\mathbf{a}$.

$$A = \gamma(R + I_3)^{-1}(R - I_3)$$

and

$$\begin{aligned} \mathbf{u} &= (1/2)\gamma\mathbf{t} - (1/2)\mathbf{a} \times \mathbf{t} \\ &= \gamma(R + I_3)^{-1}\mathbf{t} \end{aligned}$$

γ not important so,

$$\begin{pmatrix} A & \mathbf{u} \\ 0 & 0 \end{pmatrix} = \mathrm{Cay}_4^{-1}\begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}$$

Another Cayley map on $SE(3)$

Unlike exponential map, Cayley map depends on which representation of the group is used. Get a different Cayley map using the 6×6 adjoint rep.

Group elements, $H = \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}$,

T 3×3 anti-symmetric matrix corresponding to $\mathbf{t}$.

Lie algebra elements, $\mathrm{ad}(\mathbf{s}) = \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$,

A and B both 3×3 anti-symmetric.

Another Cayley map (cont.)

Definition

$$\mathrm{Cay}_6(\mathrm{ad}(\mathbf{s})) = (I_6 - \mathrm{ad}(\mathbf{s}))^{-1}(I_6 + \mathrm{ad}(\mathbf{s}))$$

can show that,

$$\mathrm{Cay}_6(\mathrm{ad}(\mathbf{s})) = \begin{pmatrix} \mathrm{Cay}(A) & 0 \\ 2(I_3 - A)^{-1}B(I_3 - A)^{-1} & \mathrm{Cay}(A) \end{pmatrix}$$

That is,

$$T = 2(I_3 - A)^{-1}B(I_3 + A)^{-1}$$

Another Cayley map (cont.)

This map really is different from Cay_4 defined above. Can see this by comparing the translational parts of the screws produced by the inverse map.
as we saw,

$$\mathbf{u} = \mathbf{q} \times \mathbf{a} + \frac{(\theta/2)}{\tan(\theta/2)} \left(\frac{p}{2\pi}\right) \mathbf{a}$$

for Cay_6^{-1} can show that,

$$\mathbf{b} = \mathbf{q} \times \mathbf{a} + \frac{\theta}{\sin\theta} \left(\frac{p}{2\pi}\right) \mathbf{a}$$

Conclusions

1. Cayley map depends on the representation, get different Cayley maps for 4×4 and 6×6 representations. Other representations possible.

Conclusions

2. Connection between Cayley maps and line complexes. Suppose there is a map that associates a line complex to each element of $SE(3)$ then this determines a map from the group to its Lie algebra. Works in other direction too, a map from the group to its Lie algebra also determines a line complex for each group element.

Conclusions

3. The Cayley map is rational, the exponential map is only analytic. This makes it very useful for numerical methods since there will be fewer transcendental function calls. For example, can be used to solving rigid body dynamics problems and drawing differentiable curves by solving Frenet-Serret equations.