



## Risk and Uncertainty in the Patent Race: a Probabilistic Model

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# Risk and Uncertainty in the Patent Race: a Probabilistic Model

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## Abstract

This article develops a novel probabilistic approach to evaluate -to be intended as provide an approximation formula- a patent-protected *R&D* project at a fixed date under very general conditions. In a real option framework we introduce Spatial Mixed Poisson Processes to describe the dynamics of the project value. In such a fashion, the model is able to account for competition among firms and several sources of uncertainty such as time-to-completion of the project, exogenous shocks, input cost uncertainty, technical uncertainty and asymmetric information under different cost structures. The proposed evaluation procedure is of the Bayesian type, in that it moves from the knowledge of some information collected before the evaluation date.

**Keywords:** Patent race; Spatial Mixed Poisson Process; real options, Bayesian estimation; uncertainty.

**JEL Classification:** C02, O32.

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# 1 Introduction

Optimal decision-making for taking out a patent is a complex task, exacerbated by the presence of competitiveness and uncertainty. These two characterizing features are intrinsically related, reciprocally entangled. Competitiveness is a source of uncertainty and the latter feeds into the former making it tougher. Indeed, in a competing context the decision maker has to take into account not only the factors that affect his own strategies, but also the factors affecting other investors' decisions. Uncertainty about others' actions -on the one side- urges taking out the patent in order to avoid being cut out by preemption. On the other side, there are situations in which it may be convenient to wait for the others' move. An example of the latter situation is provided by the millions of vaccine doses for H1N1 flu unsold, or unused and resold from developed to developing countries in 2010. In this case, it would have been optimal to wait for the others' action, at the risk of losing the first mover's benefit.

This paper deals with the basic step of this decision process, which consists in estimating the project value. More specifically, the main contribution consists in proposing a **method to approximate the expected value of the patent at a fixed date**. The procedure we implement to obtain such a result is based on the information collected before the evaluation date, and in this sense it may be viewed as of the Bayesian type.

Patent race, namely the issue of valuing an *R&D* project finalized to take out a patent in a competing and uncertain environment, has extensively been treated in the economic literature (Kamien and Schwartz, 1972; Loury, 1979; Dasgupta and Stiglitz, 1980; Fudenberg et al., 1983; Harris and Vickers, 1985, 1987; Beath et al., 1989; Nti, 1997; Weeds, 2002; Grishagin et al., 2001; Lambrecht and Perraudin, 2003; Miltersen and Schwartz, 2004; Pawlina and Kort, 2006; Meng, 2008; Pennings and Sereno, 2011; just to cite a few). In spite of the importance that uncertainty plays in patenting and more in general in *R&D*-intensive firms activity, the theoretical literature is scant in modelling and capturing it. Some authors explicitly mention the existence of multiple sources of uncertainty (Grishagin et al. 2001; Miltersen and Schwartz, 2004; Meng, 2008), but no papers have succeeded in capturing them at once. Different works focus on different sources of uncertainty, according to the specific issue the paper wants to examine in depth.

At its very essence, the multiplicity of uncertainties can be traced back to the long investment time horizon of patenting activity and more generally of *R&D*. Uncertainty about the time-to-completion magnifies the other sources of uncertainty, the longer and the more uncertain the *R&D* phase is, the riskier the project becomes.

In the literature one can find different, and partially overlapping, definitions of the sources of uncertainty. The most cited are: cash flow uncertainty, technical uncertainty, exogenous shocks, input cost uncertainty and competitive uncertainty.

Because different sources of uncertainty interact with each other, leaving some of them out of the

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3 formal scheme may be misleading in terms of project valuation.

4 The model we are going to present in the following sections accounts for all the sources of uncertainty  
5 aforementioned and the firms are assumed to operate in a competitive environment. Encompassing  
6 all the sources of uncertainty in a unique model is quite difficult, because it greatly complicates the  
7 algebra, but the task is worth tackling given the importance that uncertainty plays in *R&D* man-  
8 agement and ultimately in policy intervention. Within the real options framework, the task can be  
9 accomplished by abandoning the continuous time framework, typical of patent race models, and by  
10 using a particular type of discrete process, the Spatial Mixed Poisson Process (SMPP). This process  
11 generalizes the standard Spatial Processes derived by jumps occurring according to a Poisson Law.  
12 The adoption of random point processes in applied works is well-acknowledged in the literature. We  
13 refer to Volf (2009), Syamsundar and Naikan (2009) and Jacobsen (2006) for a survey of theory and  
14 applications. Concerning the general theory of spatial point processes, a complete survey can be  
15 found in Daley and Vere-Jones (1988) and Stoyan et al. (1995). While, for the case of SMPPs, we  
16 also refer to Cerqueti et al. (2009), where the development of an insurance-reinsurance model is  
17 provided.

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19 In our context, from an economic point of view, the use of SMPP allows to overcome some restric-  
20 tive hypothesis, typical of *R&D* valuation literature. In particular, first of all, it allows tackling the  
21 problem of determining the value of a project related to a patent in the presence of jumps in the un-  
22 derlying asset with negative and positive sizes. **Secondly, thinking of a negative jump due to**  
23 **other firms attainment, it can happen, and indeed it does, that the novelty contained in**  
24 **the others' patent is not so crucial, being, let us say, an invention around attainment.**<sup>1</sup>  
25 **This aspect is particularly relevant, in that it captures the fact that negative events can**  
26 **engender heterogenous impacts on the patent value according to the events themselves,**  
27 **but also to the timing of the event. For instance, the same event will likely impact**  
28 **differently on the value of the patent according to whether it occurs before or after the**  
29 **proclamation of a new patent-protecting law or of a new fiscal regulation. In general,**  
30 **the patent under valuation suffers from a negative jump in its value, but this may however not be**  
31 **large enough to make the patent worthless. An original feature of our approach is that the amount**  
32 **of the jumps is not known beforehand. To sum up, the *killing jump* is not restricted to the first**  
33 **negative jump that may occur, but only a *large enough* jump can kill the patenting activity. Thirdly,**  
34 **there is a random delay in the transmission of the jump from the stochastic process representing**  
35 **a given source of uncertainty to the cash flow generated by the project. This plays an important**  
36 **role in modeling random processes with different jump behaviors. It has never been clarified in the**  
37 **existing literature why other firms' attainment should bring about an immediate obsolescence in the**

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52 <sup>1</sup>By invention around it is commonly meant an invention which achieves the same or similar functions using a  
53 different means and not violating the claims of the original patent.  
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3 patent that is being valued. For instance, in the consumer electronics field it takes time before a  
4 new product is marketed and, once it is, it takes time before it is sufficiently widespread to cut to  
5 nil the profit deriving from the "old" patent. In such a case, the assumption of a no-delay condition  
6 would be a naïve approximation. Fourthly, we accommodate the case of the winner-takes-all as a  
7 special case of the winner-does-not-take-all game. Fifthly, exogenous shocks are supposed to impact  
8 with different intensities on the competing firms. Finally, the adoption of SMPP allows to take into  
9 account the dependence between time and size of the jumps in the underlying security. We will turn  
10 to these points in more details below.

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12 The rest of the paper is organized as follows. The next Section outlines the basic problem motivat-  
13 ing this research. Section 3 is devoted to the construction of a theoretical framework, taking into  
14 account multiple sources of uncertainty. IN addition, this Section contains some preliminary results,  
15 along with the formal definition of SMPP. Section 4 concerns the technical conditions stopping the  
16 patent race. A Bayesian estimate of the value of the project is derived in Section 5 and its economic  
17 implications are discussed in Section 6. Finally, Section 7 concludes and outlines future research.  
18 Some mathematical derivations are provided in the Appendix.

## 26 2 The basic problem

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28 Consider a new technology developed by firm  $\mathcal{A}$  that can possibly be monopolized by taking out a  
29 patent. Throughout the paper we will refer to  $\bar{P}$  as the patent, or project  $\bar{P}$ , or simply the project.  
30 Let us suppose the economic environment to be populated by other  $N$  competitive firms investing  
31 in  $R\&D$  to develop new technologies, including the one related to  $\bar{P}$ . We adopt the point of view  
32 of firm  $\mathcal{A}$  and we seek a theoretical estimate of the expected value of the project. In doing so, a  
33 real option type approach is followed<sup>2</sup>. The patent race problem is strictly connected to the study  
34 of the optimal time for taking out the patent  $\bar{P}$ . Clearly, the patent is attainable after the *full*  
35 *development* of the related technology, let us say at the random time  $\tau_1$ , representing uncertainty in  
36 time-to-completion. It follows that the problem of optimal timing for patent protection is considered  
37 **only when** the firm accomplishes the  $R\&D$  phase and is ready to register the patent  $\bar{P}$ .

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39 At its very essence, the structure of the model works as follows. In accordance with the real options  
40 literature, the cash flow is taken as the underlying security, or state. The changes in the cash  
41 flow immediately affect the value of the derivative, namely the project. In similarity with financial  
42 models, the real options literature treats the value of the project as a derivative, which means that  
43 it is supposed to be a function of its underlying security. The value of the project does not coincide  
44 with the value of the cash flow in that the firms have forward looking expectations. Typically,

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52 <sup>2</sup>Notice that by the words "theoretical estimate" we refer to the fact, that being the jumps in the underlying  
53 security stochastic in time and size and being the transmission time of these jumps stochastic itself, it is necessary to  
54 estimate the expected value of the impact of the jumps on patenting activity.

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3 *R&D* investments do not generate cash flow in an early stage, but they unfold their productive  
4 effects after a (long) period. In turn, this fact does not imply that the project is worthless in the  
5 investment period, as rational economic agents take into account future returns and they know that  
6 even if the project does not immediately generate cash flow, it will do in the future. The cash flow is  
7 supposed to be composed of two parts: a regular flow deriving from the scheduled outflows and an  
8 extraordinary flow, subject to stochastic fluctuations due to the economic context and competition.  
9 More specifically, some events occur during the race at discrete times. These events are captured by  
10 the jumps of  $N + 1$  distinct stochastic point processes. One of these represents the shocks occurring  
11 in the economy, e.g. changes in regulation, demand shifts, taste variations, etc. The remaining  $N$   
12 processes represent the progresses of the other  $N$  firms taking part in the race. The random jumps  
13 occurring in the  $N + 1$  processes affect  $\mathcal{A}$ 's cash flow with a certain random intensity and at random  
14 time. We will come back to the details of the transmission mechanism which determines the sizes of  
15 the jumps in Section 3. For the time being, we highlight that the time at which the events occur in  
16 the  $N + 1$  processes are not necessarily the same time at which the cash flow is affected. Indeed, the  
17 jumps in the  $N + 1$  processes do not immediately propagate to  $\mathcal{A}$ 's cash flow, but only after a certain  
18 random delay. Moreover, also the  $N$  players different from  $\mathcal{A}$  are affected by exogenous shocks. It is  
19 worth recalling that the jumps in the cash flow propagate immediately to the value of the project,  
20 without any delay. Since the cash flow jumps with a certain delay with respect to the occurrence of  
21 the event, the jumps in the cash flow that will occur "tomorrow", are partially due to the jumps in  
22 the  $N + 1$  stochastic processes occurred "today". Hence, to value the patenting activity in a future  
23 date one must give an estimate of the jumps in the  $N + 1$  processes already occurred, the effect of  
24 which will be exerted in the future.

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26 Summarizing: the underlying security,  $\mathcal{A}$ 's cash flow, is affected by exogenous shocks and by the  
27 events affecting the other  $N$  players. In turn, the latter are affected by exogenous shocks as well.  
28 Hence, exogenous shocks feed into the underlying security twice, directly and indirectly via the  
29 competitors. The transmission of the shocks is not restricted to be immediate and one to one, but  
30 occurs with random delay and intensity. Figure 1 graphically depicts this composite situation.

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INSERT FIGURE 1 ABOUT HERE

**Caption:** The jumps occurring in the economy affect the firm's cash flow with a random delay.

The delayed transformations are captured by the arrays in the graph.

The cash flow process is such to account for uncertainty of cash flow itself as well as technical uncertainty. The process describing exogenous shocks accounts also for input cost uncertainty. The random processes, by modeling competition, account for uncertainty coming from a competitive environment and asymmetric information under an asymmetric cost structure. The context is assumed to be one in which the winner does not necessarily take all.

## 2.1 The formalization of the problem.

To formalize the model, let us introduce a probability space with filtration  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ , containing all the random variables introduced in the following. We denote as  $\mathcal{T}$  the set of stopping times in  $[0, +\infty)$ , i.e.

$$\mathcal{T} = \{\tau : \Omega \rightarrow [0, +\infty) \mid \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}. \quad (1)$$

To model time-to-completion uncertainty, let us define the random time at which the new technology is fully developed by  $\mathcal{A}$  as  $\tau_1 \in \mathcal{T}$ . The optimal timing problem can be rephrased in searching for the random time  $\tau_2$  that is optimal for taking out  $\bar{P}$  and such that  $\tau_2(\omega) \geq \tau_1(\omega)$ , for each  $\omega \in \Omega$ . Fix  $t > 0$  and denote as  $C_t$  the random value of the project  $\bar{P}$  for  $\mathcal{A}$  at time  $t$ . The quantity  $C_t$  is random in that it depends on the future events concerning the life of the firms involved in the patent race and on the economic environment.

The patent is assumed to be optimally taken out when the expected value of  $C_t$  reaches a certain time-dependent deterministic threshold  $\Gamma_t$ . More formally, let us introduce the difference process

$$d_t = \Gamma_t - \mathbb{E}[C_t], \quad (2)$$

where  $\mathbb{E}$  indicates the expected value operator. Thus, the stopping time  $\tau_2$  can be written as follows:

$$\tau_2 = \begin{cases} \inf\{t \geq \tau_1 \mid d_t \leq 0\}, & \text{if } \mathbb{E}[d_{\tau_1}] > 0; \\ \tau_1, & \text{if } \mathbb{E}[d_{\tau_1}] \leq 0. \end{cases} \quad (3)$$

Of course  $\mathcal{A}$  must compete with several other firms, which try to obtain the registration of the same or similar patents. Therefore,  $\tau_1$  or  $\tau_2$  may also be equal to  $+\infty$  when the patent race ends in favor of a firm different from  $\mathcal{A}$ . We will label these situations as *game over conditions*, and they will be discussed in Section 4.

The optimization problem related to  $\tau_2$  in (3) depends on two components: the time-dependent threshold  $\Gamma_t$  and the expected value of the project  $\mathbb{E}[C_t]$ . This paper proposes a Bayesian method for estimating  $\mathbb{E}[C_t]$  in a very general framework. Since the estimation procedure is quite technical, we have chosen to focus only on this issue, without devoting much efforts neither to the identification of  $\Gamma_t$ , which can be exogenously or endogenously determined, nor to the optimization problem. Nevertheless, the results obtained hereafter will set up the basis for a follow up study coping with the optimization problem. For the time being, tackling the optimization problem exceedingly complicates and lengthens the paper.

## 3 The sources of uncertainty and the underlying asset

The theoretical analysis of patent race interactions is given through the introduction of quantitative processes translating qualitative occurrences. This approach has been already used in the literature,

and the idea of a random process transforming qualitative events into quantitative amounts is first attributable to Pindyck (1993).<sup>3</sup>

### 3.1 Overview of the main elements of the model

This subsection introduces the main components of the patent race problem. Since the argument is rather technical, we prefer to treat separately the conditions granting the existence of competition among firms (the so-called *game over conditions*, see the next section). For the time being it will be assumed that the patent race never stops.

Let us introduce a stochastic process  $T$  describing the evolution of  $A$ 's cash flow, directly related to  $\bar{P}$ . We also assume that all the sources of uncertainty impact on the cash flow generated by the  $R\&D$  investment project. It follows that  $T$  collects  $R\&D$  expenses, the inflows and the outflows stemming from the events in the economic environment and the competition with the other  $N$  firms populating the market. A formal definition of  $T$  will be presented below.

The time-dependent value of  $\bar{P}$  for  $\mathcal{A}$  can be explicitly written by collecting information on the process  $T$ . For this reason  $T$  represents the underlying asset.

To formalize the sources of uncertainty and analyze their interaction rules, a multivariate point process is needed, with  $N + 1$  components given by: an exogenous process  $S$  that captures the quantitative translation of the shocks occurring in the economic system, including input cost uncertainty; a stochastic process  $U_k$  describing the cash flows of the  $k$ -th firm's  $R\&D$  process, with  $k = 1, \dots, N$ . All the sources of uncertainty affect  $\mathcal{A}$ 's cash flow (see Figure 2 for a representation of such reticular relationships).

INSERT FIGURE 2 ABOUT HERE

**Caption:** the arrows point out that the realizations of  $T$  are driven by the evolution of the multivariate process  $(S, U_1, \dots, U_N)$  and, furthermore,  $U_k$  depends on  $S$ , for each  $k$ .

From  $\mathcal{A}$ 's point of view,  $U_k$  represents a proxy for the speed of the technological renewal process and the interest in new technologies of the  $k$ -th firm involved in the patent race.  $U_k$  captures uncertainty arising from a competitive environment and asymmetric information under an asymmetric cost structure. In such a way, besides uncertainty arising from time-to-completion, we manage to capture uncertainty due to exogenous shocks,  $S$ , technical uncertainty (or entity of  $R\&D$  expenses, or cost-to-completion), competitive interactions, asymmetric information and finally investment cost asymmetry. All these components impact differently on  $U_k$  and are somehow reflected on  $\mathcal{A}$ 's cash flow. Modelling separately the dynamics of each competitor's cash flow,  $U_k$ , we have the highest possible degree of asymmetry and uncertainty.

<sup>3</sup>In his setting a Geometric Brownian Motion is assumed to capture the shocks in the economy, which feed into the underlying state dynamics, the industry's inverse demand curve. The underlying state determines the value of the option to invest.



### 3.2 Exogenous shocks in the economy: process $S$ .

The exogenous shocks in the economy are captured by a process  $S = \{(\tau_i^S, \xi_i^S)\}_{i \in \mathbb{N}}$ , where  $\tau_i^S$  and  $\xi_i^S$  represent the random time and size of the  $i$ -th shock, for each  $i \in \mathbb{N}$ , respectively. In particular,  $\tau_i^S \in \mathcal{T}$ ,  $\xi_i^S \in \mathcal{F}_{\tau_i^S}$  and  $\xi_i^S$  takes on values in  $\mathbb{R}$ .

The random variables  $\xi_i^S$  are supposed i.i.d. and independent from  $\{\tau_i^S\}_{i \in \mathbb{N}}$ . Moreover, the following key assumption on the stochastic structure of  $S$  will stand in force hereafter.

**Assumption 1.** *The process  $S$  is a SMPP with mixing distribution  $\Pi$  and baseline intensity measure  $M$ .*

For a formal definition of SMPP, see Subsection 3.5.

An economic explanation of Assumption 1 is in order.

The process  $S$  captures also the component of uncertainty pertaining to the so called catastrophic events, that are related to negative jumps (McDonald and Siegel, 1986; Miltersten and Schwartz, 2004; Schwartz, 2004). However, hypothesizing only negative jumps to occur would be an unrealistic assumption. Indeed, there are many events which could have positive influence, e.g. a more favorable patent-protection law, more effective or rigorous enforcement policies, a complementary discovery accelerating patent attainment, a reduction in patenting cost, a reduction in the renewal fee schedule, etc. In addition, it is unrealistic to assume that the size of the jump is predetermined (in this respect see Weeds, 2002; Lambrect Perraudin, 2003). Abandoning the assumption that  $S$  follows a Poisson process, typical in the literature, and allowing it to follow a SMPP, we easily overcome both assumptions.<sup>4</sup> In this framework, jumps are not restricted to take on only negative values. Being  $\xi_i^S$  random, the size of the jump is unknown *a priori*. As a consequence, it is possible to generalize the effect of a negative jump and remove the restriction that any negative jump, in particular the first, can kill the patenting activity. Negative jumps "small" in size will induce a drawback, but not necessarily put an end to the activity. This characteristic is particularly important for *R&D* activity characterized by high entry and exit sunk costs. *R&D* cannot be costlessly suspended and resumed. Think of the cost of firing the researchers and the legally required termination payments. In this case the Marshallian rule of negative Net Present Values (NPV) to shut down the activity does not hold (Dixit, 1989). The presence of exit sunk costs requires the NPV to be greatly negative before shutting down the plant. Therefore, in this context negative but not fatal jumps are quite likely to occur and the SMPP accommodates for this feature.

For further explanations on the reasonability of Assumption 1, we refer to Subsection 3.5.

<sup>4</sup>The SMPP assumption is not strictly necessary to allow the process to take on positive jumps, but it is essential to introduce random sizes and derive our theoretical estimation result.

### 3.3 Competitors' cash flow: processes $U_k$ .

The sequence of couples  $U_k = (\tau_i^{U_k}, \xi_i^{U_k})$  formalizes the process describing times and sizes of the cash flow of the  $k$ -th firm, with  $k = 1, \dots, N$ , as they are perceived by  $\mathcal{A}$ . The process  $U_k$  collects the  $k$ -th firm's cash flow associated to the development of new technologies having the chance to take out new patents, including  $\bar{P}$ . It can be considered as a proxy for the  $R\&D$  progress of the  $N$  firms playing the race. The randomness in  $U_k$  also allows to capture the asymmetric information context, in that  $\mathcal{A}$  cannot perfectly forecast the realizations of  $U_k$ . The dependence between  $S$  and  $U_k$  is conveniently modeled as follows.

Fix  $k = 1, \dots, N$ . Define  $W_k = \{(w_{1k}^{(i)}, w_{2k}^{(i)}, w_{3k}^{(i)})\}_{i \in \mathbb{N}}$ , which are assumed to be i.i.d. and independent from  $S$  and take values in  $\mathcal{W}_{U_k} = [0, +\infty)^2 \times \mathbb{R}$ , and define a transformation

$$\phi_{U_k} : [0, +\infty) \times \mathbb{R} \times \mathcal{W}_{U_k} \rightarrow [0, +\infty) \times \mathbb{R}$$

such that

$$\phi_{U_k}(\tau_i^S, \xi_i^S, w_{1k}^{(i)}, w_{2k}^{(i)}, w_{3k}^{(i)}) = (\tau_i^S + w_{1k}^{(i)}, w_{2k}^{(i)} \xi_i^S e^{w_{3k}^{(i)} \tau_i^S}), \quad i \in \mathbb{N}. \quad (4)$$

$w_{1k}^{(i)}, w_{2k}^{(i)}, w_{3k}^{(i)}$  are random variables as in (7), and represent a stochastic delay, a stochastic scale factor and a random growth factor, respectively.

By defining

$$\tau_i^{U_k} = \tau_i^S + w_{1k}^{(i)}, \quad \xi_i^{U_k} = w_{2k}^{(i)} \xi_i^S e^{w_{3k}^{(i)} \tau_i^S} \quad \forall i \in \mathbb{N}, \quad (5)$$

we obtain a representation of the process  $U_k$ .

### 3.4 Cash flow: process $T$ .

$T$  is the process followed by  $\mathcal{A}$ 's cash flow and represents the underlying asset of the proposed real option model. In the Introduction, we have seen that given the long time profile of research projects, revisions to investment plans are very common in patenting activity. Therefore, without loss of generality, we can assume that the  $R\&D$  process involves a *regular flow* and an *extraordinary flow*.

The regular flow aims at capturing the deterministic expenses sustained periodically by  $\mathcal{A}$  to increase  $R\&D$ . It is composed of a sequence of annuities occurring at fixed dates. Spector and Zuckerman (1997) report that various  $R\&D$  projects are developed at a constant effort, in order to avoid the high cost associated with hiring and firing of highly qualified research workers. The regular flow captures this characteristic of  $R\&D$ , allowing also for varying scheduled expenses. **However, not all expenses** can be known in advance, so that a random component still remains. In this respect, the extraordinary flow captures the random amounts paid out at random times, driven by the shocks occurring in the economic context. Finally, the extraordinary flow is associated to inflows and outflows connected to the project, and it is meant to capture technical uncertainty and input

cost uncertainty. Hence, the sources of the realizations of the extraordinary flow are the processes  $S$  and  $U_k$ , with  $k = 1, \dots, N$ . Accordingly, we split the extraordinary flow in  $N + 1$  processes  $T_S, T_1, \dots, T_N$ , that are related to the jumps of  $T$  due to  $S, U_1, \dots, U_N$ , respectively.

Formally, the process  $T$  is a sum of  $N + 2$  terms:

$$T = T_O + T_S + \sum_{k=1}^N T_k,$$

where  $T_O = \{(u_i, z_i)\}_{i \in \mathbb{N}}$  is the regular flow, with  $u_i \geq 0$  and  $z_i \leq 0$  respectively representing the deterministic time and size of the  $i$ -th jump in the ordinary expenses. It is worth recalling that the process  $T_O$  stops when the technology related to  $\bar{P}$  is fully developed, at time  $\tau_1$ . From that point onwards  $\mathcal{A}$  should choose the optimal time  $\tau_2$  for taking out the patent. Formally, the threshold  $\tau_1$  -if ever reached- can be viewed as an upper bound to the time of the jumps occurring in  $T_O$ .

$T_S, T_1, \dots, T_N$  are the point processes describing the extraordinary flow, dependent on the processes  $S, U_1, \dots, U_N$ . It will turn out useful to denote as  $T^E$  the extraordinary flow of the process  $T$ , i.e.

$$T^E = T_S + \sum_{k=1}^N T_k. \tag{6}$$

The process  $T_S = \{(\tau_i^{T_S}, \xi_i^{T_S})\}_{i \in \mathbb{N}}$  can be described by introducing a sequence of trivariate i.i.d. random variables

$$W_{T_S} = \{(w_1^{(i)}, w_2^{(i)}, w_3^{(i)})\}_{i \in \mathbb{N}},$$

which are assumed to be independent from  $S$  and take values in  $\mathcal{W}_{T_S} = [0, +\infty)^2 \times \mathbb{R}$ , and a transformation

$$\phi_{T_S} : [0, +\infty) \times \mathbb{R} \times \mathcal{W}_{T_S} \rightarrow [0, +\infty) \times \mathbb{R}$$

such that

$$\phi_{T_S}(\tau_i^S, \xi_i^S, w_1^{(i)}, w_2^{(i)}, w_3^{(i)}) = (\tau_i^S + w_1^{(i)}, w_2^{(i)} \xi_i^S e^{w_3^{(i)} \tau_i^S}), \quad i \in \mathbb{N}. \tag{7}$$

By defining

$$\tau_i^{T_S} = \tau_i^S + w_1^{(i)}, \quad \xi_i^{T_S} = w_2^{(i)} \xi_i^S e^{w_3^{(i)} \tau_i^S} \quad \forall i \in \mathbb{N}, \tag{8}$$

we have a representation of the process  $T_S$  associated to the jumps of the process  $S$ .

$w_1^{(i)}, w_2^{(i)}, w_3^{(i)}$  are random variables representing a stochastic delay, a stochastic scale factor and a random growth factor, respectively. They translate times and sizes of the jumps of the exogenous process  $S$  in terms of times and sizes of  $\mathcal{A}$ 's cash flow. The exponential term in the expression of  $\xi_i^{T_S}$  captures the evidence that the effect on the value of the project of an occurrence in the economic environment is also time dependent.

The random quantities  $\tau_i^{T_S}$  and  $\xi_i^{T_S}$  are the time and size of the  $i$ -th jump in the cash flow process for  $\mathcal{A}$ , generated by the  $i$ -th event in the economic framework, respectively. In (8), the functional form of the random time,  $\tau_i^{T_S} = \tau_i^S + w_1^{(i)}$ , is such to accommodate situations in which there is a

1  
 2  
 3 jump in the exogenous process that induces a jump in the cash flow at a later random time. Think of  
 4 an environmental regulation forcing the firms to invest in expensive "scrubbers" or to buy tradable  
 5 "allowances" that allow them to pollute, a sort of Clean Air Act. Most of the time the regulation  
 6 does not force the firms to comply immediately with the new standard, letting the firm deciding  
 7 when it is optimal to delay the green investment within a reasonable amount of time. In other words,  
 8 there is a random delay between the jump in the exogenous process  $S$  and the related jumps in  $T$ .  
 9  
 10 The functional form of the size of  $T$ ,  $\xi_i^{TS} = w_2^{(i)} \xi_i^S e^{w_3^{(i)} \tau_i^S}$ , is such to accommodate the ambiguous  
 11 relationship between *good/bad news* (Baudry and Dumont, 2006) and a jump in the cash flow. A  
 12 straightforward example of *good news* is one that makes the total cost decrease, e.g. a reduction in  
 13 input cost for any reason, but also *good news* that make the firm more eager to invest are conceiv-  
 14 able. If a new regulation requiring clean energy adoption is enacted while the firm  $\mathcal{A}$  is investing  
 15 in more efficient solar panels, the firm will be spurred to finish the  $R\&D$  phase as soon as possible,  
 16 in order to commercialize the new product. For this reason, the effect of the news described by the  
 17 random jump  $\xi_i^S$  is reduced or amplified by a random factor  $w_2^{(i)}$ , which can take values in  $(0, 1)$   
 18 (case of reduction) or in  $(1, +\infty)$  (case of amplification). When  $w_2^{(i)} = 0$ , the news  $\xi_i^S$  has no impact  
 19 on the  $R\&D$  process. The random variable  $w_3^{(i)}$  is also responsible for the amplification/reduction  
 20 of the importance of the news in  $\mathcal{A}$ 's cash flows, but in this case such a term is time dependent.  
 21 For instance, a demand increase in the future patented good clearly makes the cash flow jump in  
 22 different ways when the  $R\&D$  phase is just at the beginning or when the patent is going to be  
 23 obtained in a short period of time. Yet, think of the case of a new drug against a flu developed at  
 24 the beginning or at the end of a flu epidemic course.

25  
 26 Now, fix  $k = 1, \dots, N$ , define the process  $T_k = \{(\tau_i^{T_k}, \xi_i^{T_k})\}_{i \in \mathbb{N}}$  and denote  $U_k = \{(\tau_i^{U_k}, \xi_i^{U_k})\}_{i \in \mathbb{N}}$ . To  
 27 understand the nature of the link between  $T$  and  $U_k$ , some explanations are in order.

28  
 29 In a competitive economic framework different types of managerial strategies can be adopted. One  
 30 of the most natural ones is for  $\mathcal{A}$  to establish economic relationships with some of the other firms  
 31 populating the market. In doing so, the firms cluster in two families:  $\mathcal{A}$ 's partners and  $\mathcal{A}$ 's rivals.  
 32 We then suppose the existence of  $N_p$  partners and  $N_r = N - N_p$  rivals. Now, the process  $T_k$  has to  
 33 be defined by taking into account this distinction. Without loss of generality, we can order the  $N$   
 34 partners such that  $1, \dots, N_p$  are  $\mathcal{A}$ 's partners, while  $N_p + 1, \dots, N$  are rivals.

35  
 36 Partnership or competitiveness are key issues to study the effect produced by a jump in  $U_k$  on  $\mathcal{A}$ 's  
 37 cash flow. Moreover, in a complex economic system, several technologies and related patents are  
 38 under development at the same time. Therefore, a jump in the  $k$ -th firm's  $R\&D$  process may induce  
 39 different effects on the value of  $\bar{P}$  for  $\mathcal{A}$ , depending on the relationship between  $\mathcal{A}$  and the  $k$ -th  
 40 firm, as well as on the degree of similarity between patents. In order to clarify the latter point  
 41 we introduce an index  $\alpha_k^{(i)} \in [0, 1]$ , representing a normalized measure of the distance between the  
 42 technology contained in the patent under scrutiny,  $\bar{P}$ , and the one related to the  $i$ -th jump in  $U_k$ .

We assume  $\alpha_k^{(i_1)} < \alpha_k^{(i_2)}$  when the technology implemented for  $\bar{P}$  is closer to that related to the  $i_2$ -th jump than to the one related to the  $i_1$ -th jump. The limit cases are  $\alpha_k^{(i)} = 0$ , occurring when the  $i$ -th jump has nothing to do with  $\bar{P}$ , and  $\alpha_k^{(i)} = 1$ , occurring when the  $k$ -th firm takes out patent  $\bar{P}$ . Such an index enables also to consider the interconnections between different sectors in the economy. That is, firms operating in the same sector are more likely to compete on very similar projects. If the  $k$ -th firm operates in the same sector of  $\mathcal{A}$ , then the index  $\alpha_k^{(i)}$  exhibits a high value for each  $i$ . Therefore, it is more likely that if a firm, say the  $k$ -th one, operating in the same sector of  $\mathcal{A}$  takes out a patent, we will have a value of  $\alpha_k^{(i)}$  not far from 1. **The case of a small value of  $\alpha_k^{(i)}$**  is more likely to occur when the  $k$ -th firm operates in different sectors from  $\mathcal{A}$ 's, as its investments aim at different goals. Moreover, the presence of the index  $\alpha_k^{(i)}$  makes the patent race quite general in the sense that it may be either a winner-takes-all game, when  $\alpha_k^{(i)} = 1$ , or a winner-does-not-take-all game, when  $\alpha_k^{(i)} \in [0, 1)$ .

Let us turn now to the relationship between  $U_k$  and  $T$  for  $\mathcal{A}$ 's rivals or partners. We analyze first the case of partnership,  $k \in \{1, \dots, N_p\}$ .

Define the sequence of trivariate i.i.d. random variables  $Y_{P,k} = \{(y_{1k}^{(i)}, y_{2k}^{(i)}, y_{3k}^{(i)})\}_{i \in \mathbb{N}}$ , which are assumed to be independent from  $S$  and take values in  $\mathcal{Y}_{P,k} = [0, +\infty)^2 \times \mathbb{R}$ , and define a transformation

$$\phi_{P,k} : [0, +\infty) \times \mathbb{R} \times \mathcal{Y}_{P,k} \rightarrow [0, +\infty) \times \mathbb{R}$$

such that

$$\phi_{P,k}(\tau_i^{U_k}, \xi_i^{U_k}, y_{1k}^{(i)}, y_{2k}^{(i)}, y_{3k}^{(i)}) = (\tau_i^{U_k} + y_{1k}^{(i)}, (1 - e^{-\alpha_k^{(i)}})[y_{2k}^{(i)} \xi_i^{U_k} e^{y_{3k}^{(i)} \tau_i^{U_k}}]), \quad i \in \mathbb{N}. \quad (9)$$

The relation between  $U_k$  and  $T$  in this case is formalized by defining

$$\tau_i^{T_k} = \tau_i^{U_k} + y_{1k}^{(i)}, \quad \xi_i^{T_k} = (1 - e^{-\alpha_k^{(i)}})[y_{2k}^{(i)} \xi_i^{U_k} e^{y_{3k}^{(i)} \tau_i^{U_k}}] \quad \forall i \in \mathbb{N}. \quad (10)$$

$y_{1k}^{(i)}$  is a random delay,  $y_{2k}^{(i)}$  is a random scale factor and  $y_{3k}^{(i)}$  is a random growth factor.

**The definition of  $\phi_{P,k}$**  is in accordance with that of  $\phi_{T_S}$  in (7), with a couple of key differences. A random delay appears, due to the possible delay in the propagation of information from  $k$  to  $\mathcal{A}$ . A term related to  $\alpha_k^{(i)}$  is introduced, in order to describe the dependence of  $\xi_i^{T_k}$  on the type of technology implemented. If a new attainment is made by a partner of  $\mathcal{A}$ , and such an attainment is complementary to  $\bar{P}$  -in the sense that it helps the attainment of  $\bar{P}$ - then  $\mathcal{A}$  benefits from this jump, and the value of the patent will increase. In this respect, we notice that the function

$$h : [0, 1] \rightarrow [0, 1] \quad | \quad h(\alpha_k^{(i)}) = 1 - e^{-\alpha_k^{(i)}} \quad (11)$$

is increasing with respect to  $\alpha_k^{(i)}$  and null when  $\alpha_k^{(i)} = 0$ . Hence,  $h$  in (11) is a suitable choice for capturing the dependence between jumps and developed technologies. The fact that the function  $h(\cdot)$  at its maximum does not reach unity for  $\alpha_k^{(i)} = 1$  is irrelevant because it is multiplied by a

random scale factor in (9),  $y_{2k}^{(i)}$ , which magnifies or dampens the impact of  $\xi_i^{U_k}$  on  $\xi_i^{T_k}$ . Again,  $y_{3k}^{(i)}$  is a random growth factor, and it formalizes the time dependence, as explained in the discussion of  $\phi_{T_S}$ .

Let us now assume that  $k \in \{N_p + 1, \dots, N\}$ , i.e.  $k$  is  $\mathcal{A}$ 's rival.

The relationship between  $T$  and  $U_k$ , with  $k$  being  $\mathcal{A}$ 's rival is similar to that between  $T$  and  $U_k$  with  $k$  partner, but with an opposite sign.

Define the sequence of trivariate i.i.d. random variables  $Z_{R,k} = \{(z_{1k}^{(i)}, z_{2k}^{(i)}, z_{3k}^{(i)})\}_{i \in \mathbb{N}}$ , which are assumed to be independent from  $S$  and take values in  $\mathcal{Z}_{R,k} = [0, +\infty)^2 \times \mathbb{R}$ , and introduce a transformation

$$\phi_{R,k} : [0, +\infty) \times \mathbb{R} \times \mathcal{Z}_{R,k} \rightarrow [0, +\infty) \times \mathbb{R}$$

such that

$$\phi_{R,k}(\tau_i^{U_k}, \xi_i^{U_k}, z_{1k}^{(i)}, z_{2k}^{(i)}, z_{3k}^{(i)}) = (\tau_i^{U_k} + z_{1k}^{(i)}, (e^{-\alpha_k^{(i)}} - 1)[z_{2k}^{(i)} \xi_i^{U_k} e^{z_{3k}^{(i)} \tau_i^{U_k}}]), \quad i \in \mathbb{N}. \quad (12)$$

As usual, we formalize the relationship between  $U_k$  and  $T$  by assuming:

$$\tau_i^{T_k} = \tau_i^{U_k} + z_{1k}^{(i)}, \quad \xi_i^{T_k} = (e^{-\alpha_k^{(i)}} - 1)[z_{2k}^{(i)} \xi_i^{U_k} e^{z_{3k}^{(i)} \tau_i^{U_k}}] \quad \forall i \in \mathbb{N}. \quad (13)$$

$z_{1k}^{(i)}$  is a random delay,  $z_{2k}^{(i)}$  is a random scale factor and  $z_{3k}^{(i)}$  is a random growth factor.

In the rest of the paper, we refer to  $T = (\tau_i^T, \xi_i^T)$  when referring to the process  $T$  without considering which process drives the jumps.

### 3.5 Preliminary comments and results

Table (1) schematically reports the relationships between the source of uncertainty and the process capturing it.

Table 1: Type of uncertainty captured by the process

Process	Type of uncertainty captured by the process
cash flow, $T$	cash flow, technical uncertainty (cost-to-completion)
Exogenous shock, $S$	macroeconomic shocks, catastrophic events, change in laws, etc.)
other firms' cash flow, $U_k$	competitive environment, asymmetric informations, asymmetric cost structure

It is worth noting that the same sort of transmission mechanism for the exogenous shocks on  $\mathcal{A}$  applies to the other  $N$  firms. That is, the value of the  $N$  firms' projects are supposed to be random and similar to  $\mathcal{A}$ 's, although not perfectly identical because, in general,

$$(w_{1k}^{(i)}, w_{2k}^{(i)}, w_{3k}^{(i)}) \neq (w_{1k_2}^{(i)}, w_{2k_2}^{(i)}, w_{3k_2}^{(i)}), \quad \forall k = 1, \dots, N, \quad (14)$$

and

$$(w_{1k_1}^{(i)}, w_{2k_1}^{(i)}, w_{3k_1}^{(i)}) \neq (w_{1k_2}^{(i)}, w_{2k_2}^{(i)}, w_{3k_2}^{(i)}), \quad k_1 \neq k_2. \quad (15)$$

Beyond the obvious statement on different effects of the same event on firms pursuing different new technologies, conditions (14) and (15) meet also the evidence of asymmetric cost structure among firms belonging to the same or different sectors. The choice of investment cost asymmetry can be motivated by a number of reasons (Pawlina and Kort, 2006). First of all, investment cost asymmetry is present when firms have different access to capital markets. The cost of capital of a firm facing liquidity constraints is higher than that of its counterpart having access to a credit line or with substantial cash reserves (Lensink et al., 2001). Furthermore, cost asymmetry occurs when the firms exhibit a different degree of organizational flexibility in implementing a new production technology. This flexibility, known as absorptive capacity, measures the firm's ability to adopt external technologies, to adapt to a changing economic environment, and to commercialize newly invented products. A higher absorptive capacity is therefore equivalent to a lower cost associated with an investment project (Cohen and Levital, 1994). Yet, the difference in investment costs can be justified on the ground of purely exogenous factors, resulting, among others, from the intervention of the authority. For instance, the effective investment cost of the firms is reduced after obtaining governmental credit guarantee, which results in a lower cost of capital (evidence can be found in Klemeier and Megginson, 2000; Zecchini and Ventura, 2009).

By a purely mathematical point of view, formulas (7) and (4) show that there is a correlation structure between  $\xi_i^{T_S}$  and  $(\tau_i^S, \xi_i^S)$  as well as between  $\xi_i^{U_k}$  and  $(\tau_i^S, \xi_i^S)$ . This fact does not allow to consider  $S$  as a stochastic process on the line, and we need to treat  $S$  as a Spatial Point Process. In this respect, the benefits of assuming  $S$  to be a SMPP are twofold. First of all, it provides a model for simultaneously estimating the number and the size of the jumps in the economy and, consequently, the number and the size of the cash flows of the  $N + 1$  firms populating the market, including  $\mathcal{A}$ , and related to  $\bar{P}$ . Secondly, according to some recent results, SMPPs can guarantee the invariance of the stochastic structure between  $S$ ,  $T_S$  and  $U_k$ , for each  $k$  (see Theorem 1). The same argument applies to the correlation structure between  $\xi_i^{T_k}$  and  $(\tau_i^{U_k}, \xi_i^{U_k})$ , shown in (9) and (12).

In order to be self-contained, we introduce the formal definition of SMPP.

Let us consider a measure space  $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), M)$ , where  $\mathcal{B}(\mathbb{R}^k)$  is the Borel  $\sigma$ -algebra and  $M$  is absolutely continuous with respect to the Lebesgue measure. We also introduce a nonnegative random variable  $\Lambda$  with probability distribution  $\Pi : \mathbb{R} \rightarrow [0, 1]$ .

**Definition 1.** *A spatial process  $S$  is Mixed Poisson with mixing distribution  $\Pi$  and baseline intensity measure  $M(\cdot)$  if and only if, for  $I \in \mathcal{B}(\mathbb{R}^k)$  and for  $n \in \mathbb{N}$ ,*

$$P(S(I) = n) = \int_0^\infty e^{-\lambda M(I)} \frac{[\lambda M(I)]^n}{n!} d\Pi(\lambda). \quad (16)$$

We now need an important invariant result, already known in the literature (see e.g. Cinlar (1995)) and a proof based on stochastic geometrical arguments can be found in Foschi and Spizzichino

(2008).

**Theorem 1.** *Let Assumption 1 be satisfied and consider a sequence of i.i.d. random variables  $\mathbf{W} = \{W_i\}_{i \in \mathbb{N}}$ , with distribution  $G$  and independent of  $S$ .*

*Then  $T = \Phi_\phi(S, \mathbf{W})$  is a SMPP with the same mixing distribution  $\Pi$  and intensity measure*

$$M^*(J) = \int_{\mathbb{R}^n} M(\phi_w^{-1}(J)) dG(w), \quad (17)$$

*where  $J \subseteq \mathcal{Y}$  and  $X \in \phi_w^{-1}(J)$  if and only if  $\phi(X, w) \in J$ .*

Theorem 1 is a key result in our work, since it explains the invariance of SMPPs with respect to a very general class of transformations. As a consequence, we can write:

**Proposition 1.** *Let Assumption 1 be satisfied. Then  $T_S, T_1, \dots, T_N$  are SMPPs, with the same mixing distribution and baseline intensity measures  $M^{(S)}, M^{(1)}, \dots, M^{(N)}$  as in (17), respectively.*

It is worth noting that, even if the sizes of the stochastic jumps are theoretically unbounded, we argue that the cash flow of a firm may not be infinite. Similarly the quantitative process capturing the entity of the news appearing in the economic environment should be bounded, at least from below, by the game over conditions. Thus, without loss of generality, one can reasonably assume the existence of positive upper and negative lower thresholds for the stochastic processes describing uncertainty in the underlying asset. Therefore, hereafter we assume the existence of the positive constants  $a_S, a_T, a_1, \dots, a_N$  such that, fixed  $i \in \mathbb{N}$ ,  $\xi_i^S, \xi_i^T, \xi_i^{U_1}, \dots, \xi_i^{U_N}$  are random variables with support in  $[-a_S, a_S], [-a_T, a_T], [-a_1, a_1], \dots, [-a_N, a_N]$ , respectively.

## 4 The game over conditions

In a patent race context, firm  $\mathcal{A}$  loses the competition in two cases:

- (i) when  $\mathcal{A}$  abandons the project to develop the technology associated to  $\bar{P}$ ;
- (ii) when the technology has been already developed by  $\mathcal{A}$ , but someone else registers the patent  $\bar{P}$  before  $\mathcal{A}$ .

The cases (i) and (ii) correspond to a certain jump in  $\mathcal{A}$ 's cash flow, which has been named in the Introduction as *killing jump*. In both cases, we achieve a situation where the cash flow process  $T$  stops, and the value of the project remains permanently below the threshold  $\Gamma$  in (2).

To model this occurrence, we need to introduce a time dependent threshold:

$$\Theta : [0, +\infty) \rightarrow [0, +\infty) \quad | \quad t \mapsto \Theta(t) \quad (18)$$

such that  $\Theta$  *kills* the value of the patent, e.g.

$$\xi_i^T \leq -\Theta(\tau_i^T) \iff \xi_j^T = 0, \quad \forall j > i.$$



If the threshold  $\Theta$  is reached, then the process  $T$  stops and firm  $\mathcal{A}$  loses the competition. The occurrence of a killing jump may depend on a *very bad news*, the so-called catastrophic event, originated in the economic environment, and captured by a jump in the process  $S$ , such as the situation where the patentable drug turns out to have terrible side effects. In this respect, there exists a time dependent threshold

$$\gamma : [0, +\infty) \rightarrow [0, +\infty) \quad | \quad t \mapsto \gamma_t \tag{19}$$

such that

$$\xi_i^{Ts} \leq -\Theta(\tau_i^{Ts}), \quad \text{for } \xi_i^S \leq -\gamma_{\tau_i^S}. \tag{20}$$

A killing jump may also depend on the registration of the patent  $\bar{P}$  by another firm. The consequences in the long-run of this event differ according to the nature of the relationship between  $\mathcal{A}$  and the winning firm, i.e. partnership or rivalry. However, the analysis of the future exploitation of the technology related to  $\bar{P}$  is out of the scope of this paper. We can generally say that the registration of  $\bar{P}$  by the  $k$ -th firm is the end of the patent race, for each  $k = 1, \dots, N$ . This fact involves the relationship between the processes  $U_k$  and  $T$ , and is associated only to the case of  $\alpha_k^{(i)} = 1$ .

Analogously to  $\gamma_t$  defined in (19), there exists a time dependent threshold

$$\psi : [0, +\infty) \rightarrow [0, +\infty) \quad | \quad t \mapsto \psi_t \tag{21}$$

such that

$$\xi_i^{Tk} \leq -\Theta(\tau_i^{Tk}), \quad \text{for } \xi_i^{Uk} \leq -\psi_{\tau_i^{Uk}}. \tag{22}$$

## 5 The value of the project

As clearly explained by Pakes and Simpson (1989) and Shankerman and Pakes (1986), fixed  $t \geq 0$ ,  $C_t$  can be regarded as the value at time  $t$  of the project associated to the patent  $\bar{P}$ . Actually,  $C_t$  is a random variable in that it sums up also the discounted future random cash flow of  $\mathcal{A}$ , and the estimate of the patent value will be pursued through the computation of its conditioned expected value.

In details, consistently with the definition of  $T$  and formula (6), we write  $C_t$  as the sum of the contributions due to the ordinary and extraordinary flows or, more formally, as the discounted sum of the processes  $T_0, T_S, T_1, \dots, T_N$ :

$$C_t = C_t^{To} + C_t^{TE}, \tag{23}$$

where

$$C_t^{TE} = C_t^{Ts} + \sum_{k=1}^N C_t^{Tk}, \tag{24}$$

and the superscripts indicate the reference processes. By (24), the terms in the right-hand side of (23) can be written as:

$$\begin{cases} C_t^{To} = \sum_{i=1}^{+\infty} z_i \beta^{u_i-t} \mathbf{1}_{\{t \leq \tau_1\}} \\ C_t^{TS} = \sum_{i=1}^{+\infty} \xi_i^{TS} \beta^{\tau_i^{TS}-t} \\ C_t^{Tk} = \sum_{i=1}^{+\infty} \xi_i^{Tk} \beta^{\tau_i^{Tk}-t} \end{cases} \quad (25)$$

where  $\beta \in (0, 1)$  is an appropriate discount factor. By inserting the game over conditions and by formula (25), (23) becomes:

$$C_t = \sum_{i=1}^{+\infty} \left[ z_i \beta^{u_i-t} \mathbf{1}_{\{t \leq u_i \leq \tau_1\}} + \xi_i^{TS} \beta^{\tau_i^{TS}-t} + \sum_{k=1}^N \xi_i^{Tk} \beta^{\tau_i^{Tk}-t} \right] \cdot \mathbf{1}_{\{\xi_i^{TS} > -\Theta(\tau_i^{TS})\}} \cap \left[ \bigcap_{k=1}^N \{\xi_i^{Tk} > -\Theta(\tau_i^{Tk})\} \cap \{\alpha_k^{(i)} \neq 1\} \right] \quad (26)$$

A Bayesian-type mechanism for the estimation of the expected value of  $C_t^{TE}$  in (26) is now provided, following the approach of Cerqueti et al. (2009).

Define a time interval

$$\mathcal{I} = [r_1, r_2]$$

with  $0 < r_1 < r_2$ . The interval  $\mathcal{I}$  represents the period *under scrutiny*, and constitutes the basic set of the Bayesian estimate.

The shocks in the economic environment become delayed jumps of the process  $T^E$ . Therefore, if we consider the jumps in  $S$  at times  $\tau_i^S \in \mathcal{I}$ , then the corresponding jumps of the process  $T^E$  may occur in  $\mathcal{I}$  or after  $r_2$ .

We now define the regions  $I_S \equiv \mathcal{I} \times [-a_S, a_S]$ ,  $I_T \equiv \mathcal{I} \times [-a_T, a_T]$  and  $I_k \equiv \mathcal{I} \times [-a_k, a_k]$ , for  $k = 1, \dots, N$ .

In agreement with a commonly used mathematical notation, we introduce  $D \subseteq \mathbb{R}^2$  and denote by  $S(D)$ ,  $T(D)$ ,  $U_1(D), \dots, U_N(D)$  the number of the elements of the Spatial Point Process  $S$ ,  $T^E$ ,  $U_1, \dots, U_N$  respectively, that are contained in  $D$ .

Being the SMPPs simple processes (see Cerqueti et al., 2009) and being regions  $I_S, I_T, I_1, \dots, I_N$  bounded, then  $\mathbb{E}[S(I_S)] < +\infty$ ,  $\mathbb{E}[T^E(I_T)] < +\infty$  and  $\mathbb{E}[U_k(I_k)] < +\infty$ , for each  $k = 1, \dots, N$ .

Moreover, the Bayesian-type mechanism is based on the knowledge of the cash flows of  $\mathcal{A}$  related to  $\bar{P}$  and the jumps in the economic environment occurring over the time interval under scrutiny  $\mathcal{I}$ .

Define the time interval  $\mathcal{H} = [t, +\infty)$ , with  $t > r_2$ , and the region  $H = \mathcal{H} \times [-a_T, a_T]$ .

**Remark 1.** We will assume hereafter that  $E[T(H)] < +\infty$ . As a supporting argument, we can say that the cash flows of  $\mathcal{A}$  related to  $\bar{P}$  are measured in units of currency, and so the future cash flows cannot be unbounded, even if they are aggregated.

The conditional expectation of  $C_t^{TE}$  will be approximated by the knowledge of the following quantities:

- the number  $S(I_S)$  of jumps in the economy during the interval  $\mathcal{I}$ ;
- the number  $T^E(I_T)$  of the jumps in the cash flow of  $\mathcal{A}$  during  $\mathcal{I}$ . Such a quantity is the aggregation of the contributions of the processes  $T_S, T_1, \dots, T_N$ , i.e.:

$$T^E(I_T) = T_S(I_S) + \sum_{k=1}^N T_k(I_k);$$

- the number of jumps in the economy occurred over  $\mathcal{I}$  propagated to  $\mathcal{A}$ 's cash flow in the same time interval. We denote this quantity as  $T_{(I_S)}^E(I_T)$ <sup>5</sup>, and it is given by the aggregation of the contributions of the processes  $T_S, T_1, \dots, T_N$ , i.e.:

$$T_{(I_S)}^E(I_T) = T_{S(I_S)}(I_S) + \sum_{k=1}^N T_{k(I_S)}(I_k).$$

We consider two partitions  $\Delta$  and  $\Psi$  of  $H$  as follows:

$$\Delta_j = \{H_r^{(j)}\}_{r=1, \dots, j}, \quad j \in \mathbb{N}, \tag{27}$$

where  $H_r^{(j)} = \mathcal{H} \times (c_{r-1}^{(j)}, c_r^{(j)}]$ , with  $r = 1, \dots, j$ ,  $c_0^{(j)} = -a_T$ ,  $c_j^{(j)} = a_T$  and, for each  $j$ ,  $\{c_r^{(j)}\}$  is increasing with respect to  $r$ ;

$$\Psi_h = \{G_v^{(h)}\}_{v=1, \dots, h}, \quad h \in \mathbb{N}, \tag{28}$$

where  $G_v^{(h)} = [t_{v-1}^{(h)}, t_v^{(h)}) \times [-a_T, a_T]$ , with  $v = 1, \dots, h$ ,  $t_0^{(h)} = t$ ,  $t_h^{(h)} = +\infty$  and, for each  $h$ ,  $\{t_v^{(h)}\}$  is increasing with respect to  $v$ .

A further refined partition of  $H$  can be obtained by the intersection of the partitions defined in (27) and (28). We have

$$\Delta_j \cap \Psi_h = \left\{ H_r^{(j)} \cap G_v^{(h)} \right\}_{\substack{v \leq h; \\ r \leq j}}, \quad j, h \in \mathbb{N}. \tag{29}$$

Now, define the event

$$A(\mathcal{I}) = \left\{ S(I_S) = p_S, T_S(I_S) = n_S, T_1(I_1) = n_1, \dots, T_N(I_N) = n_N, \right. \\ \left. T_{S(I_S)}(I_S) = m_S, T_{1(I_S)}(I_1) = m_1, \dots, T_{N(I_S)}(I_N) = m_N \right\}. \tag{30}$$

We fix the four integers  $r, v, h, j$  and denote by  $b_{r,v}^{(j,h)}$  the expected number of jumps modeling  $\mathcal{A}$ 's cash flow observed in the time interval  $(t_{v-1}^{(h)}, t_v^{(h)})$  with size  $(c_{r-1}^{(j)}, c_r^{(j)})$ , conditioned on the previous history in period  $\mathcal{I}$ . i.e.

$$b_{r,v}^{(j,h)} \equiv \mathbb{E} \left[ T(H_r^{(j)} \cap G_v^{(h)}) \mid A(\mathcal{I}) \right]. \tag{31}$$

The next result provides a closed form expression to compute  $b_{r,v}^{(j,h)}$ , for any  $(j, h) \in \mathbb{N}^2$  and  $(r, v) \in \{1, \dots, j\} \times \{1, \dots, h\}$ . To proceed, following the definition of the event  $A(\mathcal{I})$  in (30), a distinction between the sources of the jumps of  $T^E$  is needed.

<sup>5</sup>The interested reader can find a formal derivation of  $T_{(I_S)}^E(I_T)$  in the Appendix.

**Proposition 2.**

$$b_{r,v}^{(j,h)} = \sum_{\bar{k} \in \{S, 1, \dots, N\}} \left[ \sum_{n=0}^{+\infty} \sum_{l=0}^n n \cdot \Gamma_1(\bar{k}) \cdot \Gamma_2(\bar{k}) \cdot \Gamma_3(\bar{k}) \cdot \Gamma_4(\bar{k}) \right], \quad (32)$$

with

$$\left\{ \begin{array}{l} \Gamma_1(\bar{k}) = \frac{[M_{(I_S)}^{(\bar{k})}(H_r^{(j)} \cap G_v^{(h)})]^{n-l}}{(n-l)!} \\ \Gamma_2(\bar{k}) = \frac{[M_{(I_S)}^{(\bar{k})}(H_r^{(j)} \cap G_v^{(h)})]^l}{l!} \\ \Gamma_3(\bar{k}) = \int_0^{+\infty} \lambda^{n-l} e^{-\lambda M_{(I_S)}^{(\bar{k})}(H_r^{(j)} \cap G_v^{(h)})} u(\lambda; I_S, I_{\bar{k}}, p_S, n_{\bar{k}}, m_{\bar{k}}) d\lambda \\ \Gamma_4(\bar{k}) = \int_0^{+\infty} \lambda^l e^{-\lambda M_{(I_S)}^{(\bar{k})}(H_r^{(j)} \cap G_v^{(h)})} u(\lambda; I_S, I_{\bar{k}}, p_S, n_{\bar{k}}, m_{\bar{k}}) d\lambda. \end{array} \right. \quad (33)$$

where  $M_{(I_S)}^{(\bar{k})}$  is the baseline intensity measure of the SMPP  $T_{\bar{k}(I_S)}$  and

$$u(\lambda; I_S, I_{\bar{k}}, p_S, n_{\bar{k}}, m_{\bar{k}}) = \frac{\lambda^{n_{\bar{k}} - m_{\bar{k}} + p_S} e^{-\lambda [M_{(I_S)}^{(\bar{k})}(I_{\bar{k}}) + M^{(\bar{k})}(I_S)]} u(\lambda)}{\int_0^{+\infty} \lambda^{n_{\bar{k}} - m_{\bar{k}} + p_S} e^{-\lambda [M_{(I_S)}^{(\bar{k})}(I_{\bar{k}}) + M^{(\bar{k})}(I_S)]} u(\lambda) d\lambda} \quad (34)$$

is the posterior distribution of  $\Lambda$ .

*Proof.* First of all, we notice that

$$b_{s,v}^{(j,h)} \equiv \sum_{\bar{k}=S,1,\dots,N} \mathbb{E} \left[ T_{\bar{k}}(H_s^{(j)} \cap G_v^{(h)}) \mid S(I_S) = p_S, T_{\bar{k}}(I_T) = n_{\bar{k}}, T_{\bar{k}(I_S)}(I_T) = m_{\bar{k}} \right]. \quad (35)$$

Therefore, it is sufficient to prove that, for a given  $\bar{k} \in \{S, 1, \dots, N\}$ , we have

$$\mathbb{E} \left[ T_{\bar{k}}(H_s^{(j)} \cap G_v^{(h)}) \mid S(I_S) = p_S, T_{\bar{k}}(I_T) = n_{\bar{k}}, T_{\bar{k}(I_S)}(I_T) = m_{\bar{k}} \right] = \sum_{n=0}^{+\infty} \sum_{l=0}^n n \cdot \Gamma_1(\bar{k}) \cdot \Gamma_2(\bar{k}) \cdot \Gamma_3(\bar{k}) \cdot \Gamma_4(\bar{k}). \quad (36)$$

We can write:

$$\begin{aligned} & P(T_{\bar{k}}(H_s^{(j)} \cap G_v^{(h)}) = n \mid S(I_S) = p_S, T_{\bar{k}}(I_T) = n_{\bar{k}}, T_{\bar{k}(I_S)}(I_T) = m_{\bar{k}}) \\ &= \sum_{l=0}^{+\infty} P(T_{\bar{k}}(H_s^{(j)} \cap G_v^{(h)}) = n \mid S(I_S) = p_S, T_{\bar{k}}(I_T) = n_{\bar{k}}, T_{\bar{k}(I_S)}(I_T) = m_{\bar{k}}, T_{\bar{k}(I_S)}(H_s^{(j)} \cap G_v^{(h)}) = l) \times \\ & \quad \times P(T_{\bar{k}(I_S)}(H_s^{(j)} \cap G_v^{(h)}) = l \mid S(I_S) = p_S, T_{\bar{k}}(I_T) = n_{\bar{k}}, T_{\bar{k}(I_S)}(I_T) = m_{\bar{k}}), \end{aligned} \quad (37)$$

where the conditioning on the event  $T_{\bar{k}(I_S)}(H_s^{(j)} \cap G_v^{(h)}) = l$  has been removed by summing on the index  $l$ .

Therefore, by Theorem 2 of Cerqueti et al. (2009), we can rewrite (37) as follows:

$$P(T_{\bar{k}}(H_s^{(j)} \cap G_v^{(h)}) = n \mid S(I_S) = p_S, T_{\bar{k}}(I_T) = n_{\bar{k}}, T_{\bar{k}(I_S)}(I_T) = m_{\bar{k}}) =$$

$$= \sum_{l=0}^n \Gamma_1(\bar{k}) \cdot \Gamma_2(\bar{k}) \cdot \Gamma_3(\bar{k}) \cdot \Gamma_4(\bar{k}). \tag{38}$$

Since

$$\begin{aligned} & \mathbb{E} \left[ T_{\bar{k}}(H_s^{(j)} \cap G_v^{(h)}) = n \mid S(I_S) = p_S, T_{\bar{k}}(I_T) = n_{\bar{k}}, T_{\bar{k}(I_S)}(I_T) = m_{\bar{k}} \right] = \\ & = \sum_{n=0}^{+\infty} n \cdot P(T_{\bar{k}}(H_s^{(j)} \cap G_v^{(h)}) = n \mid S(I_S) = p_S, T_{\bar{k}}(I_T) = n_{\bar{k}}, T_{\bar{k}(I_S)}(I_T) = m_{\bar{k}}), \end{aligned} \tag{39}$$

and by (38), the result is proved.  $\square$

Proposition 2 allows to get an upper and a lower approximation of  $\mathbb{E}[C_t^{T^E} \mid A(\mathcal{I})]$ .

Consider the following sequences:

$$\begin{aligned} \bar{\phi}_{j,h} &= \sum_{r=1}^j \sum_{v=1}^h c_r^{(j)} \beta^{t-t_{v-1}^{(h)}} b_{r,v}^{(j,h)}, \\ \hat{\phi}_{j,h} &= \sum_{r=1}^j \sum_{v=1}^h c_{r-1}^{(j)} \beta^{t-t_v^{(h)}} b_{r,v}^{(j,h)}. \end{aligned}$$

For any  $j, h \in \mathbb{N}$ , we have

$$\hat{\phi}_{j,h} \leq \mathbb{E}[C_t^{T^E} \mid A] \leq \bar{\phi}_{j,h}. \tag{40}$$

$\{\hat{\phi}_{j,h}\}$  is non-decreasing and  $\{\bar{\phi}_{j,h}\}$  is non-increasing with respect to  $j$  and  $h$ .

We now conveniently thicken the decomposition  $\Delta_j \cap \Psi_h$  in (29). Assume that

$$\lim_{j \rightarrow +\infty} c_r^{(j)} - c_{r-1}^{(j)} = 0, \quad \forall r \in \{1, \dots, j\}, \tag{41}$$

and

$$c_0^{(j)} = -a_T \quad \text{and} \quad \lim_{j \rightarrow +\infty} c_j^{(j)} = a_T. \tag{42}$$

Moreover, assume that

$$\lim_{h \rightarrow +\infty} t_v^{(h)} - t_{v-1}^{(h)} = 0, \quad \forall v \in \{1, \dots, h\}, \tag{43}$$

and

$$t_0^{(h)} = t \quad \text{and} \quad \lim_{h \rightarrow +\infty} t_h^{(h)} = +\infty. \tag{44}$$

Under conditions (41-44), we have that

$$\bigcup_{j,h \in \mathbb{N}} \Delta_j \cap \Psi_h = H, \tag{45}$$

and there exists a nonnegative constant dependent on  $t$ , named  $p(t)$ , such that

$$\lim_{j,h \rightarrow +\infty} \hat{\phi}_{j,h} = \lim_{j,h \rightarrow +\infty} \bar{\phi}_{j,h} = p(t).$$

$p(t)$  represents the Bayesian estimate of the value of the patent accruing from the extraordinary flow. Indeed, by (40), taking the limit, we can conclude that

$$\mathbb{E}[C_t^{T^E} \mid A(\mathcal{I})] = p(t). \tag{46}$$

We now add also  $C_t^{TO}$  to  $C_t^{TE}$  of (46), and derive an estimate for  $C_t$ . By (26) we have:

$$\begin{aligned} \mathbb{E}[C_t | A] &= p(t) + \mathbb{E} \left\{ \sum_{i=1}^{+\infty} z_i \beta^{u_i - t} \mathbf{1}_{\{t \leq u_i \leq \tau_1\}} \cdot \mathbf{1}_{\{\xi_i^{TS} > -\Theta(\tau_i^{TS})\}} \cap \left[ \bigcap_{k=1}^N (\{\xi_i^{Tk} > -\Theta(\tau_i^{Tk})\} \cap \{\alpha_k^{(i)} \neq 1\}) \right] \right\} = \\ &= p(t) + \sum_{i=1}^{+\infty} z_i \beta^{u_i - t} \mathbf{1}_{\{t \leq u_i\}} P(u_i \leq \tau_1) \cdot P \left( \left\{ \xi_i^{TS} > -\Theta(\tau_i^{TS}) \right\} \cap \left[ \bigcap_{k=1}^N \left( \{\xi_i^{Tk} > -\Theta(\tau_i^{Tk})\} \cap \{\alpha_k^{(i)} \neq 1\} \right) \right] \right). \end{aligned} \quad (47)$$

In order to solve (47) we need to compute the probabilities involved in it. Denote by  $F_1$  the probability distribution of the random variable  $\tau_1$ . Then:

$$P(u_i \leq \tau_1) = 1 - F_1(u_i). \quad (48)$$

The variable  $\alpha_k^{(i)}$  may be assumed discrete. We have:

$$P(\alpha_k^{(i)} \neq 1) = 1 - P(\alpha_k^{(i)} = 1). \quad (49)$$

Define the time dependent set  $I_\Theta(t) \subseteq \mathbb{R}^2$  as follows:

$$I_\Theta(t) = \{(s, \zeta(s)) \in [t, +\infty) \times \mathbb{R} \mid \zeta(s) \leq -\Theta(s)\}, \quad \forall t \geq 0. \quad (50)$$

The game over conditions, obtained counting the jumps such that  $P(\xi_i^{TS} > -\Theta(\tau_i^{TS}))$  and  $P(\xi_i^{Tk} > -\Theta(\tau_i^{Tk}))$  can be expressed through the processes  $T_S$  and  $T_k$ , respectively, and the set  $I_\Theta(t)$  defined in (50). Let us explain this point. We develop the theory only for  $T_S$ , being the case of  $T_k$  analogous. Start with  $i = 1$  and suppose that  $\xi_1^{TS} > -\Theta(\tau_1^{TS})$ . Then go to  $i = 2$  and check if  $\xi_2^{TS} > -\Theta(\tau_2^{TS})$ . If this is the case, go to  $i = 3$  and so on. The process stops at jump  $\bar{i}$ , where  $\xi_{\bar{i}}^{TS} \leq -\Theta(\tau_{\bar{i}}^{TS})$ . We then argue that the number of realizations of  $T_S$  falling in the set  $I_\Theta(t)$  becomes equals to 1 for  $i = \bar{i}$ .

Consider now the set  $\{t_v^{(h)}\}$ , with  $h \in \mathbb{N}$  and  $v = 1, \dots, h$ , introduced in (28) and assume that the conditions (43) and (44) are fulfilled. Construct a partition of  $I_\Theta(t)$  with elements defined as follows:

$$I_\Theta^v(t) = I_\Theta(t) \cap \left\{ [t_{v-1}^{(+\infty)}, t_v^{(+\infty)}) \times \mathbb{R} \right\}, \quad \forall t \geq 0 \text{ and } v \in \mathbb{N}. \quad (51)$$

Since  $T_S$  is a simple process and by condition (43) we can assume that there exists an index dependent on  $i$ , say  $v(i) \in \mathbb{N}$  such that  $\tau_i^{TS} \in [t_{v(i)-1}^{(+\infty)}, t_{v(i)}^{(+\infty)})$  and  $\tau_{i+1}^{TS} \geq t_{v(i)}^{(+\infty)}$ , for each  $i \in \mathbb{N}$ .

For the arguments developed above, we have a critical threshold associated to  $\bar{i}$ , say  $\bar{v}(\bar{i})$ , such that the number of realizations of the process  $T_S$  falling in  $I_\Theta^v(t)$  is 0, for  $v = 1, \dots, \bar{v}(\bar{i}) - 1$ , while it is 1 when  $v = \bar{v}(\bar{i})$ . Therefore, the index  $\bar{v}(\bar{i})$  can be defined as:

$$\bar{v}(\bar{i}) = \inf \left\{ v(\bar{i}) \in \mathbb{N} \text{ such that } \sum_{n=1}^{v(\bar{i})} T_S(I_\Theta^n(t)) = 1 \right\}. \quad (52)$$

We put a subscript  $S$  and  $k$  to  $v$ , to distinguish the cases of  $T_S$  and  $T_k$ . The process  $T_S$  and  $T_k$  are not independent, since they both depend on the realizations of  $S$ . Therefore:

$$P \left( \left\{ \xi_i^{TS} > -\Theta(\tau_i^{TS}) \right\} \cap \left[ \bigcap_{k=1}^N \left( \{\xi_i^{Tk} > -\Theta(\tau_i^{Tk})\} \cap \{\alpha_k^{(i)} \neq 1\} \right) \right] \right) =$$

$$\begin{aligned}
 &= [1 - P(\alpha_k^{(i)} = 1)] \times P\left(\left\{\xi_i^{T_S} > -\Theta(\tau_i^{T_S})\right\} \mid \bigcap_{k=1}^N \left\{\xi_i^{T_k} > -\Theta(\tau_i^{T_k})\right\}\right) \times \\
 &\quad \times P\left(\left\{\xi_i^{T_N} > -\Theta(\tau_i^{T_N})\right\} \mid \bigcap_{k=1}^{N-1} \left\{\xi_i^{T_k} > -\Theta(\tau_i^{T_k})\right\}\right) \times \\
 &\quad \times \dots \times P\left(\left\{\xi_i^{T_2} > -\Theta(\tau_i^{T_2})\right\} \mid \left\{\xi_i^{T_1} > -\Theta(\tau_i^{T_1})\right\}\right) \times P\left(\left\{\xi_i^{T_1} > -\Theta(\tau_i^{T_1})\right\}\right). \tag{53}
 \end{aligned}$$

Hence, by (53), the game over conditions in (47) become:

$$\begin{aligned}
 &P\left(\left\{\xi_i^{T_S} > -\Theta(\tau_i^{T_S})\right\} \cap \left[\bigcap_{k=1}^N \left(\left\{\xi_i^{T_k} > -\Theta(\tau_i^{T_k})\right\} \cap \{\alpha_i^k \neq 1\}\right)\right]\right) = \\
 &= [1 - P(\alpha_k^{(i)} = 1)] \times P\left(T_S \left(I_{\Theta}^{v_S(i)}(t) - \left[\bigcup_{k=1}^N I_{\Theta}^{v_k(i)}(t)\right]\right) = 0\right) \times \\
 &\quad \times P\left(T_N \left(I_{\Theta}^{v_N(i)}(t) - \left[\bigcup_{k=1}^{N-1} I_{\Theta}^{v_k(i)}(t)\right]\right) = 0\right) \times \\
 &\quad \times \dots \times P\left(T_2 \left(I_{\Theta}^{v_2(i)}(t) - I_{\Theta}^{v_1(i)}(t)\right) = 0\right) \times P\left(T_1 \left(I_{\Theta}^{v_1(i)}(t)\right) = 0\right). \tag{54}
 \end{aligned}$$

By applying Definition 1, condition (54) becomes:

$$\begin{aligned}
 &P\left(\left\{\xi_i^{T_S} > -\Theta(\tau_i^{T_S})\right\} \cap \left[\bigcap_{k=1}^N \left(\left\{\xi_i^{T_k} > -\Theta(\tau_i^{T_k})\right\} \cap \{\alpha_i^k \neq 1\}\right)\right]\right) = \\
 &\quad [1 - P(\alpha_k^{(i)} = 1)] \times \int_0^{\infty} e^{-\lambda M^{(S)}} \left(I_{\Theta}^{v_S(i)}(t) - \left[\bigcup_{k=1}^N I_{\Theta}^{v_k(i)}(t)\right]\right) d\Pi(\lambda) \times \\
 &\quad \times \int_0^{\infty} e^{-\lambda M^{(N)}} \left(I_{\Theta}^{v_N(i)}(t) - \left[\bigcup_{k=1}^{N-1} I_{\Theta}^{v_k(i)}(t)\right]\right) d\Pi(\lambda) \times \\
 &\quad \times \dots \times \int_0^{\infty} e^{-\lambda M^{(2)}} \left(I_{\Theta}^{v_2(i)}(t) - I_{\Theta}^{v_1(i)}(t)\right) d\Pi(\lambda) \times \int_0^{\infty} e^{-\lambda M^{(1)}} \left(I_{\Theta}^{v_1(i)}(t)\right) d\Pi(\lambda). \tag{55}
 \end{aligned}$$

Denote the right-hand side of (55) as  $\Upsilon$ .

By substituting (48) and (55) into (47), we obtain:

$$\mathbb{E}[C_t | A(\mathcal{I})] = p(t) + \sum_{i=1}^{+\infty} z_i \beta^{u_i - t} \mathbf{1}_{\{t \leq t_i\}} [1 - F_1(u_i)] \times \Upsilon. \tag{56}$$

Equation (56) claims that the value of the project is given by the sum of two components. The first one,  $p(t)$ , represents the discounted expected value of the jumps in the underlying security, while the second one concerns the discounted expected value of the ordinary expenses. In this quantity, the term  $[1 - F_1(\cdot)]$  captures the presence of the stopping time  $\tau_1$  associated to the random time at which the new technology is fully developed and the term  $\Upsilon$  represents the probability that a game over condition does not occur.

## 6 Implications of our analysis.

Despite its intuitive significance and the close similarity to the results present in the literature (for instance see Miltersen and Schwartz, 2004, eq. 12) (56) reveals some critical considerations. Looking at the time profile of the estimate we can claim that,  $p(t)$  comes from the algebraic sum of two components:

$$p(t) = p_H^{(a)}(t) + p_H^{(b)}(t), \quad (57)$$

where  $p_H^{(a)}(t)$  represents the jumps in the project due to the jumps in the economy occurred in  $H$  and propagated in the same time period;  $p_H^{(b)}(t)$  represents the jumps in the project value due to the jumps in the economy occurred before  $H$  and propagated in  $H$ .

Formula (57) is useful to compare the value of the project obtained under (56) with the value of the project obtained under an alternative naive model. By naive model, we mean a model in which no delay between the economy and the patent jump is assumed. Under this restrictive hypothesis, let us denote by  $\tilde{p}_H(t)$  the aggregation of future expected discounted value of the jumps in the cash flow at time  $t$ , due to the jumps in the economy over a time interval  $H$ . The no delay condition implies that

$$\tilde{p}_H(t) = p_H^{(a)}(t), \quad (58)$$

where  $p_H^{(a)}(t)$  is defined as in (57).

By (57) and (58), we generally have that  $p(t) \neq \tilde{p}_H(t)$ , since there is no reason to expect  $p_H^{(b)}(t) = 0$ . In other words,  $p(t) = \tilde{p}_H(t)$  is a special case of (56) and its occurrence is purely accidental. Therefore, we argue that paying a little price in terms of algebraic effort, the model better fits the real world, providing us with a theoretical estimate of the value of the project. So far, the comparison between our model and an alternative naive one has been kept as simple as possible, but the divergence can be worsened if one considers also that computation of  $p(t)$  entails taking into account some remarkable features of the race that a naive approach completely neglects. Such as: the possibility of both positive and negative jumps, the possibility of non deadly jumps which are usually omitted from the models, the possibility of either the winner-taks-all or the winner-does not-take-all, the different sensitivity to exogenous shock of different firms. Put another way, the higher the uncertainty in the race, the greater the divergence between  $p(t)$  and  $\tilde{p}_H(t)$ .

A quantitative appraisal of the difference is far beyond the scope of this paper. We limit ourselves to bring to notice how neglecting some source of uncertainty can heavily bias the estimation of the project.



## 7 Conclusions and further research

The real option theory is based on the definition of an underlying asset which evolves stochastically over time and uncertainty feeds back dynamically in the value of the derivative. In this paper we have shown that by exploiting this basic idea it is possible to model patent race taking into account the multiple sources of uncertainty it implies, without requiring unrealistic or abrupt assumptions, such as winner-takes-all or winner-does-not-take-all. At the same time, we depart from the standard real option framework by considering an underlying asset evolving in discrete time as a Spatial Mixed Poisson Process. To our best knowledge, this is the first paper that models the entire set of uncertainty sources in a unique framework, providing a theoretical estimate of the value of the project. Moreover, the introduction of a distance measure between competing projects enables to account for both complementarity between innovations and winner-takes-all as possible outcomes of the race. Finally, our model allows to handle the case in which players of the race form alliances. The Bayesian estimate of the value of the project derived in this work may constitute the basis for the solution of an optimization problem regarding the optimal time of patent registration. For a discussion of this aspect see Section 2. The empirical validation of our theoretical results may represent a further challenging research theme to be explored. In this respect, it is worth noting that the numerical analysis of the proposed estimation methodology is relevant both from an economic and from a more quantitative perspective. Indeed, to the best of our knowledge, so far there are no contributions in the literature reporting a simulation of the SMPPs. Our future research efforts will focus on these topics.

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## 44 Appendix

### 45 Formal derivation of $T_{(I_S)}(I_T^*)$

46 Let us define the random subset of indexes  $\{i_1, \dots, i_{\bar{K}}\} \subset \mathbb{N}$ , such that

$$47 \{(\tau_{i_1}^S, \xi_{i_1}^S), \dots, (\tau_{i_{\bar{K}}}^S, \xi_{i_{\bar{K}}}^S)\} = I_S \cap S. \quad (59)$$

48 We notice that  $\bar{K} \equiv S(I_S)$ .

49 The game over conditions have to be taken into account. By (20), if there exists an index  $i_{\bar{K}} < i_{\bar{K}}$

in  $\{i_1, \dots, i_{\bar{K}}\}$  such that  $\xi_{i_{\bar{K}}}^S \leq -\gamma_{\tau_{i_{\bar{K}}}^S}$ , then the patent race stops and the jumps of interest in  $I_S \cap S$  are  $\{(\tau_{i_1}^S, \xi_{i_1}^S), \dots, (\tau_{i_{\bar{K}}}^S, \xi_{i_{\bar{K}}}^S)\}$ . Therefore, we can introduce  $K = \bar{K} \wedge \tilde{K}$  and rewrite (59) as:

$$\{(\tau_{i_1}^S, \xi_{i_1}^S), \dots, (\tau_{i_K}^S, \xi_{i_K}^S)\} = I_S^* \cap S, \quad (60)$$

and  $K \equiv S(I_S^*)$ . The superscript  $*$  means that the number of elements of the process  $S$  in  $I_S$  has to be intended *under the game over conditions* (20).

The processes  $U_k$  are generated by the exogenous process  $S$ , for each  $k = 1, \dots, N$ . Therefore, fixed  $k = 1, \dots, N$ , there exists a family of sets of indexes  $\{j_{k,1}, \dots, j_{k,\bar{K}_k}\} \subset \{i_1, \dots, i_K\}$ , such that:

$$\{(\tau_{j_{k,1}}^{U_k}, \xi_{j_{k,1}}^{U_k}), \dots, (\tau_{j_{k,\bar{K}_k}}^{U_k}, \xi_{j_{k,\bar{K}_k}}^{U_k})\}, \quad (61)$$

and  $\bar{K}_k \equiv U_{k(I_S^*)}(I_k)$ , where the subscript  $(I_S^*)$  means that the cash flows of the processes  $U_k$  stem from events captured by jumps of  $S$  occurring in  $I_S$  under (20).

The game over condition (22) suggests that the jumps in the processes  $U_k$  leading to a jump in the process  $T^E$  are the ones before the killing jump. More formally, if an index  $j_{k,\bar{K}_k} < j_{k,\bar{K}_k}$  can be found in  $\{j_{k,1}, \dots, j_{k,\bar{K}_k}\}$  such that  $\xi_{j_{k,\bar{K}_k}}^{U_k} \leq -\psi_{\tau_{j_{k,\bar{K}_k}}^{U_k}}$ , then the patent race stops. Analogously to what we did in (60), we then define  $K_k = \bar{K}_k \wedge \tilde{K}_k$  such that (61) can be rewritten as:

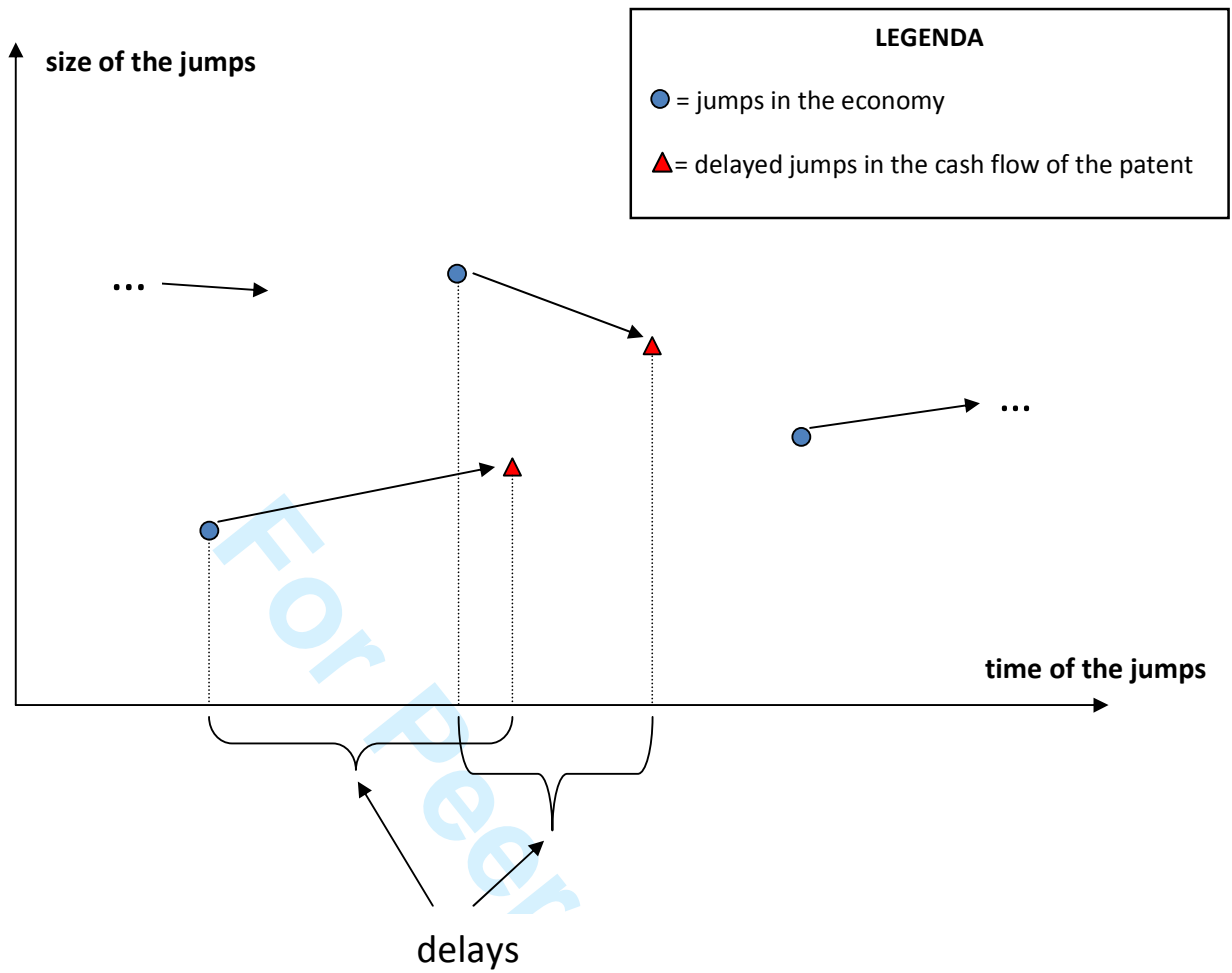
$$\{(\tau_{j_{k,1}}^{U_k}, \xi_{j_{k,1}}^{U_k}), \dots, (\tau_{j_{k,K_k}}^{U_k}, \xi_{j_{k,K_k}}^{U_k})\}, \quad (62)$$

and  $K_k \equiv U_{k(I_S^*)}(I_k^*)$ , where the subscript  $*$  on  $(I_k)$  means that the cash flows of the processes  $U_k$  are counted in  $I_k$  under condition (22). The realizations of the process  $T^E$  derive by the jumps of  $S$  and of the  $U$ 's. Following the same argument proposed above and regarding the jumps in  $T^E$  grounded on jumps of  $S$  in  $I_S$ , we can say that there exist some set of indexes  $\{l_1, \dots, l_{K_{T,S}}\} \subset \{i_1, \dots, i_K\}$  and  $\{n_{k,1}, \dots, n_{k,K_{T,k}}\} \subset \{j_{k,1}, \dots, j_{k,K_k}\}$ , for  $k = 1, \dots, N$ , such that

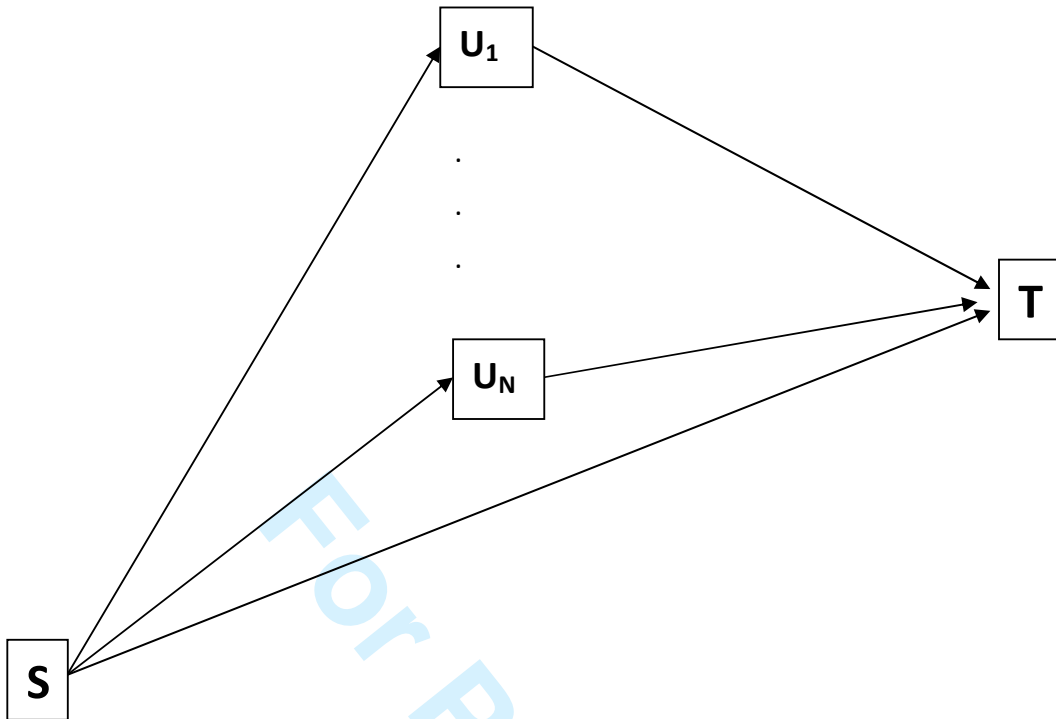
$$I_T \cap T^E = \{(\tau_{l_1}^T, \xi_{l_1}^T), \dots, (\tau_{l_{K_{T,S}}}^T, \xi_{l_{K_{T,S}}}^T)\} \cup \left[ \bigcup_{k=1}^N \{(\tau_{n_{k,1}}^T, \xi_{n_{k,1}}^T), \dots, (\tau_{n_{k,K_{T,k}}}^T, \xi_{n_{k,K_{T,k}}}^T)\} \right]. \quad (63)$$

In this case

$$T_{(I_S)}^E(I_T) \equiv K_{T,S} + \sum_{k=1}^N K_{T,k}.$$



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