Mean-variance portfolio selection in presence of unfrequently traded stocks

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Abstract

This paper deals with a mean-variance optimal portfolio selection problem in presence of risky assets characterized by low frequency of trading and, therefore, low liquidity. To model the dynamics of illiquid assets, we introduce pure-jump processes. This leads to the development of a portfolio selection model in a mixed discrete/continuous time setting. In this paper, we pursue the twofold scope of analyzing and comparing either long-term investment strategies as well as short-term trading rules. The theoretical model is analyzed by applying extensive Monte Carlo experiments, in order to provide useful insights from a financial perspective.

Keywords: Markowitz model, thin stocks, mean-variance utility function, jump-diffusion dynamics, stochastic control problem, Monte Carlo.

1 Introduction

Portfolio selection represents one of the most explored topics in finance, both from a theoretical and a practical perspective. The pioneering work on the analysis of wealth allocation is due to Markowitz (1952). This celebrated uniperiodal model moves from the basic assumptions that the return of each investment obeys a normal law and transaction costs are absent. These statements are necessary to

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grant greater mathematical tractability.

The original framework continues to attract scientific interest. Cerqueti and Spizzichino (2012) provide a rereading of the classical model of 1952 through the concept of semi-copula, and derive some specific behaviors of investors taking position in a mean-variance uniperiodal market, with normally distributed stock returns. In this respect, the most prestigious -and, maybe, most recent- reference is due to the father of portfolio selection himself, who continues to witness the debate on his original theory and has provided a survey on a half-century of research on mean–variance approximation to expected utility (see Markowitz, 2012).

Despite the relevance of the original setting, the assumptions underlying Markowitz' model seem to be too restrictive and rather unrealistic, even if they contribute to build a very nice and elegant framework of portfolio selection. Hence, the original framework has been extended in several respects by Markowitz' followers.

Some Authors propose an extension of the classical mean-variance model in a multiperiodal setting. Amongst the others, see the classical works of Samuelson (1969), Hakansson (1971) and Mossin (1968). More recently, it is worth citing Li and Ng (2000), whose work is extended by Leippold et al. (2004) that propose also a brief literature review on the mathematical difficulties generally encountered in solving mean-variance models in a multiperiodal setting.

A different strand of literature focuses on portfolio selection models in a continuous time environment. The pioneer of such an improvement is Merton (1969, 1971). In the last decade, several important contributions have appeared: Zhou and Li (2000); Lim (2004); Xia (2005); Li and Zhou (2006); Steinbach (2001) and Bielecki et al. (2005), only to cite a few. The last two papers are also very useful for the extensive discussion on the history of the original Markowitz framework.

Continuous-time models represent a remarkable improvement of Markowitz (1952), even if they are not able to capture all the peculiarities of the structure characterizing financial markets. Two paradigmatic examples are represented by transaction costs -which occur only when a transaction is performed- and liquidity deficiencies -which imply occasional trading opportunities on illiquid stocks. In particular, the stock liquidity represents a key aspect of a trading strategy. It is commonly accepted that the frequency of trade may be viewed as a measure of liquidity, in the sense that low frequency trading is synonymous of low liquidity. In the following, unfrequently traded stocks will also be denoted as *thin* or *light* stocks.

It is important to note that thin securities are rather widespread in international stock markets, mainly in emerging countries. Given the nature of unfrequently traded assets, it is also easy to guess a significant relationship between low trading volumes and low market quality (i.e.: wide bid/ask

spreads, high volatility, low informative efficiency, high adverse selection costs, etc.). In actual facts, such links have been documented by several empirical studies (see, for example, Easley et al., 1996).
In connection with the above mentioned, we propose an extension of the mean-variance Markowitz' model by considering also the presence of light stocks in the market. We deal with an optimal portfolio model in a mean-variance framework considering that the market is assumed to contain a riskless bond, frequently traded risky assets and thin stocks. We proceed as follows: first, we base the optimal portfolio selection on long-run strategies. The decision criterion is given by the maximization of the expected discounted mean-variance utility function associated to portfolio returns; second, we construct a short-term trading rule in order to provide an algorithm for deciding day-by-day a market strategy. Such a trading rule moves from the optimal portfolio found in the long-run context. A comparison between the long-run optimal portfolio and the short-term trading strategy is performed, and a related financial discussion is carried out.
The problem is faced both under a theoretical perspective as well as an extensive simulation analysis. More precisely, we first propose a rigorous mathematical formalization of the wealth allocation problem, and then study it by applying Monte Carlo simulations which allow to supply a more intuitive treatment of the proposed model.
The merging of the mean-variance multiperiodal framework with a financial setting characterized by the presence of thin stocks represents the main novelty of this paper. To the best of our knowledge, this is the first time that such a proposal appears.
By a mathematical perspective, as already argued above, the continuous-time hypothesis seems to be able to realistically describe the dynamics of frequently traded risky assets, but it becomes unreasonable when low liquidity is considered. To remove this inconsistency, we introduce random discrete trading times for the dynamics of the returns of thin stocks. In details, we model the dynamics of the light stocks through jump processes following a Levy law. In doing this, we adopt the framework discussed by Pham and Tankov (2008), Cretarola et al. (2011) and Castellano and Cerqueti (2012). It is worth to note that Levy processes are particularly appropriate for providing a distribution of the returns of unfrequently traded assets (in this respect, Schoutens (2003), and Cont and Tankov (2003), provide excellent surveys on the use of Levy processes in financial modelling).
The assumption on the dynamics of the light stocks leads to portfolios with returns evolving according to jump-diffusions. Some prominent works dealing with portfolio selection in a jump-diffusion context are: Aase (1984); Framstad et el. (1998, 2001), and Castellano and Cerqueti (2012). Matsumoto (2006) and Rogers (2001) model discrete random trading times which follow a Poisson law. Cretarola et al. (2011) deal with a consumption/portfolio selection model where only illiquid assets are con-

sidered. Each of the quoted papers and the present one face the wealth allocation problem through stochastic control theory. For an excellent tutorial on this field, in the particular case of state variables driven by jump-diffusions, see Oksendal and Sulem (2007).

The paper is organized as follows. Next section presents the description of the financial environment, along with the dynamics of the returns of the risky assets and the riskless bond available in the market. Section 3 contains the theoretical optimal portfolio selection problem in the case of long-run strategies. Section 4 concerns the description of the short-term trading rule in a very general setting. Section 5 describes the procedure used in the experiments and, in the trading rule, which is based on Sharpe ratios (see Sharpe, 1966). Last section discusses the results of the application and provide some concluding remarks.

2 The model

This section outlines the basic ingredients of the considered financial market.

We first introduce a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ containing the random quantities used in this paper, where the filtration $\mathcal{F}_t$ collects the information provided by the market up to time t.

The investor can take a financial position by investing in some assets: a risk free bond; N_L liquid risky assets and N_T thin stocks. The sets of assets are: $\mathcal{S} = \{1, 2, \ldots, i, \ldots N_L\}$ (frequently-traded assets); $\mathcal{H} = \{N_L + 1, N_L + 2, \ldots, j, \ldots, N_L + N_T\}$ (thin stocks) and $\{N_L + N_T + 1\}$ (risk-free bond).

- The price of the riskless bond D_t evolves according to the following ordinary differential equation:

$$\begin{cases} dD_t = rD_t dt, & t \geq 0, \\ D_0 = 1, \end{cases} \tag{1}$$

  where r is the deterministic continuously compounded risk free interest rate;

- the price at time t of the liquid risky assets constitutes a N_L-dimensional column vector $S_t = (S_t^i)_{i\in\mathcal{S}}$, and evolves as follows:

$$\begin{cases} dS_t = \mu_1 S_t dt + \sigma_1 S_t dW_t, & t > 0, \\ S_0 = s, \end{cases} \tag{2}$$

  where $\mu_1 = (\mu_1^i)_{i\in\mathcal{S}}$ and $\sigma_1 = (\sigma_1^i)_{i\in\mathcal{S}}$ are the column vectors of the expected rates of returns and volatilities of the liquid risky assets, respectively; W is a standard N_L-dimensional Brownian motion;

- the price at time t of the thin stocks is a N_T-dimensional column vector $H_t = (H_t^j)_{j\in\mathcal{H}}$ and it is assumed to follow a geometric Brownian motion:

$$\begin{cases} dH_t = \mu_2 H_t dt + \sigma_2 H_t dB_t, & t > 0, \\ H_0 = h, \end{cases} \tag{3}$$

  where $\mu_2 = (\mu_2^j)_{j\in\mathcal{H}}$ and $\sigma_2 = (\sigma_2^j)_{j\in\mathcal{H}}$ are the expected rates of return and the volatilities of the thin stocks, respectively; B is a standard N_T-dimensional Brownian motion.

We assume, as it should be, that $\mu_1^i > r$, for each $i \in \mathcal{S}$. By definition of thin stock, it is also realistically assumed that $\mu_2^j > \mu_1^i$ and $\sigma_2^j > \sigma_1^i$, for each $i \in \mathcal{S}$ and $j \in \mathcal{H}$.
The correlation among the assets is formalized through a symmetric variance-covariance matrix of order $N_L + N_T$, say $\Sigma = (\sigma_{h,k})_{h,k\in\mathcal{S}\cup\mathcal{H}}$, which is assumed to be constant with respect to time.
The financial characteristics of thin stocks imply that the dynamics of their returns should be modeled by a pure jump-type process. According to the model of Pham and Tankov (2008), we assume that investors can trade the j-th thin stock only at random times $\{\tau_s^j\}_{s\geq 0}$, with $\tau_0^j = 0 < \tau_1^j < \cdots < \tau_s^j < \ldots$. Only at τ's, the price of the thin stocks changes and a price jump occurs. We denote by Z_s^j the stochastic return of the j-th light stock in the random time interval (τ_{s-1}^j, τ_s^j), for each $s \in \mathbb{N}$:

$$Z_s^j = \frac{H_{\tau_s}^j - H_{\tau_{s-1}}^j}{H_{\tau_{s-1}}^j}. \tag{4}$$

Random times are assumed to be not influenced by the dynamics of the prices of liquid and light assets. To meet this evidence, we state an assumption, that will stand in force hereafter.

Assumption 2.1. *The τ's are independent from the Brownian motions W's and B's.*

The point process $\{(\tau_s^j, Z_s^j)\}_{s\in\mathbb{N}}$ is assumed to be given by the jumps of a Lévy process $\Gamma_j(t)$. We do not lose of generality by assuming that $\Gamma_j(t)$ is cadlag, which means that a jump at time t is described by $\Delta\Gamma_j(t) = \Gamma_j(t) - \Gamma_j(t-)$.

3 Long-term analysis: portfolio selection

The agent can share her/his whole capital among the assets available in the market.
Investment strategies depend on time, as it naturally should be. Moreover, in line with the original Markowitz' model, the ordered shares of capital invested in risky assets, thin stocks and riskless bond at time t are deterministic and constitute a vector $\chi(t)$ as follows:

$$\chi(t) = (\theta(t), \zeta(t), \rho(t)), \tag{5}$$

where $\theta(t) = (\theta_i(t))_{i\in\mathcal{S}}$ and $\zeta(t) = (\zeta_j(t))_{j\in\mathcal{H}}$ are the vectors of shares of capital invested at time t in the N_L risky assets and in the N_T light stocks, respectively, while $\rho(t)$ represents the share of capital invested in the riskless bond. The collection of such time-dependent quantities is a time-dependent portfolio, and will be hereafter denoted as:

$$\chi = \{\chi(t)\}_{t\geq 0}. \tag{6}$$

Moreover, since thin stocks have discrete random returns, also the ζ's should be affected by discrete times τ's. More precisely, given $t, k > 0$, $\zeta_j(t+k) \neq \zeta_j(t)$ only if there exists $s \in \mathbb{N}$ such that $t \leq \tau_s^j \leq t+k$. We will denote $\zeta_j(\tau_s^j) =: \zeta_j^s$, for $s \in \mathbb{N}$ and $j \in \mathcal{H}$.
At each time t, the capital is totally shared among the assets. Therefore:

$$\sum_{i\in\mathcal{S}} \theta_i(t) + \sum_{j\in\mathcal{H}} \zeta_j(t) + \rho(t) = 1, \qquad \forall\, t \geq 0. \tag{7}$$

The value of the portfolio χ at time $t \geq 0$, namely $X_\chi(t)$, can be written as follows:

$$X_\chi(t) = x + \int_0^t X_\chi(\xi) \left\{ \sum_{i\in\mathcal{S}} \theta_i(\xi)\mu_1^i + \rho(\xi) r \right\} d\xi +$$

$$+ \sum_{i\in\mathcal{S}} \int_0^t X_\chi(\xi)\theta_i(\xi)\sigma_1^i dW_\xi^i + \sum_{j\in\mathcal{H}} \sum_{s=1}^{+\infty} X_\chi(\tau_s^j)\zeta_j^s Z_s^j \mathbf{1}_{\{\tau_s^j \leq t\}}, \tag{8}$$

where $X_\chi(0) = x > 0$ is the initial value of the portfolio.
Given a Borel set $\mathcal{B}$ in $\mathbb{R}$, the number of jumps of the point process $\{(\tau_s^j, Z_s^j)\}_{s\in\mathbb{N}}$ occurring in the period $[0, t]$ with size in $\mathcal{B}$ can be written as follows:

$$P_j(t, \mathcal{B}) = \sum_{s=1}^{+\infty} \sharp_{\mathcal{B}}(\Delta\Gamma_j(\tau_s^j)) \mathbf{1}_{\{\tau_s^j < t\}}, \tag{9}$$

where, for each $s \in \mathbb{N}$, we define

$$\sharp_{\mathcal{B}}(\Delta\Gamma_j(\tau_s^j)) := \begin{cases} 1, & \text{if } \Delta\Gamma_j(\tau_s^j) \in \mathcal{B}; \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

We replace the discrete process Z with its continuous version Γ and rewrite (8) as follows:

$$X_\chi(t) = x + \int_0^t X_\chi(\xi) \left\{ \sum_{i\in\mathcal{S}} \theta_i(\xi)\mu_1^i + \rho(\xi) r \right\} d\xi +$$

$$+ \sum_{i\in\mathcal{S}} \int_0^t X_\chi(\xi)\theta_i(\xi)\sigma_1^i dW_\xi^i + \sum_{j\in\mathcal{H}} \int_0^t \int_{-1}^{+\infty} X_\chi(\xi)\zeta_j^\xi z P_j(d\xi, dz), \tag{11}$$

where $P_j(d\xi, dz)$ is the differential of $P_j(t, \mathcal{B})$.
The following assumption will stand in force hereafter.

Assumption 3.1. *$\{\tau_s^j\}_{s\in\mathbb{N}}$ is a sequence of jumps of a Poisson process with intensity λ_j.*

The random variable Z_l^j is independent from $\{(\tau_s^j, Z_s^j)\}_{s<l}$, being $\{(\tau_s^j, Z_s^j)\}_{s\in\mathbb{N}}$ a Levy process. We denote the distribution of Z_l^j as $p_j(t, dz)$, where $t = \tau_l^j - \tau_{l-1}^j$.

It is now worthy to recall an important result already stated in literature (see Protter, 2003, Theorem 1.35).

Theorem 3.2. *Consider a Borel set $\mathcal{B}$ in $\mathbb{R}$ and define:*

$$Y_j(t) := \sum_{s=1}^{+\infty} Z_s^j \mathbf{1}_{\{\tau_s^j \le t\}}.$$

The Levy measure ν_j of $Y_j(t)$ is given by:

$$\nu_j(\mathcal{B}) = \mathbb{E}[P_j(1, \mathcal{B})] = \lambda_j \cdot p_j(t, \mathcal{B}), \tag{12}$$

where $\mathbb{E}$ is the usual expected value operator.

The following assumption will stand in force hereafter:

Assumption 3.3.

$$\int_{-1}^{+\infty} (1 \wedge |z|)\nu_j(dz) < +\infty, \qquad \forall\, j \in \mathcal{H}. \tag{13}$$

Assumption 3.3 moves from Theorem 3.2 and implies that the jumps of each thin stock return have finite variation.

In order to end up with the formalization of the optimization problem, we introduce the definition of the return of portfolio χ as follows:

$$\Pi_\chi(t-s) = \frac{X_\chi(t) - X_\chi(s)}{X_\chi(s)}, \qquad 0 < s < t.$$

The investor aims at searching for the best investment strategy to maximize the aggregate discounted mean-variance expected utility associated to the return of her/his portfolio. The objective functional J can be defined as follows:

$$J(x, \chi) := \int_0^{+\infty} e^{-\rho t} \left[\mathbb{E}_x(\Pi_\chi(dt)) - \gamma \mathbb{V}_x(\Pi_\chi(dt))\right], \qquad x > 0, \tag{14}$$

where $\gamma > 0$ is the risk-aversion parameter, $e^{-\rho}$ is an unitary discount factor while $\mathbb{E}_x$ and $\mathbb{V}_x$ indicates the conditional expectation and variance, given $X_\chi(0) = x$, respectively.

The value function of the optimal investment problem is written as the maximization of the objective functional J as follows:

$$V(x) := \sup_{\chi \in \mathcal{A}(x)} J(x, \chi), \qquad x > 0, \tag{15}$$

where $\mathcal{A}(x)$ is the admissible region, and contains all the admissible portfolios χ as defined in (5)-(6):

$$\mathcal{A}(x) = \left\{(\theta, \zeta, \rho) : [0, +\infty)^{N_L+N_T+1} \to \mathbb{R} \,|\, \text{condition (7) holds}\right\}.$$

The problem is of convex programming-type, and it admits an unique solution $\chi^\star$ defined as follows:

$$\chi^\star = \{\chi^\star(t)\}_{t\geq 0} = \{(\theta^\star(t), \zeta^\star(t), \rho^\star(t))\}_{t\geq 0}. \tag{16}$$

4 Short-term analysis: trading rule

It may be interesting to analyze the performance of the long-run optimal portfolio in the short-term. We here aim at considering the optimal portfolio obtained in the long-run analysis and measure its performances on a restricted set of dates preceding the trading days. This represents a combination of long-run and short-term analysis.

According to (11), the value of the optimal portfolio $\chi^\star$ is:

$$X_{\chi^\star}(t) = x + \int_0^t X_{\chi^\star}(\xi) \left\{ \sum_{i\in\mathcal{S}} \theta_i^\star(\xi)\mu_1^i + \rho^\star(\xi) r \right\} d\xi +$$

$$+ \sum_{i\in\mathcal{S}} \int_0^t X_{\chi^\star}(\xi)\theta_i^\star(\xi)\sigma_1^i dW_\xi^i + \sum_{j\in\mathcal{H}} \int_0^t \int_{-1}^{+\infty} X_{\chi^\star}(\xi)(\zeta_j^\xi)^\star z N_j(d\xi, dz). \tag{17}$$

The return of portfolio $\chi^\star$ in the time interval (s,t) will be simply denoted as $\Pi^\star(t-s)$.

The short-term analysis is performed by introducing a trading rule. At this aim and without losing of generality, we assume hereafter that time is discrete as measuring *trading days*, and label it as $\{t_u\}_{u\in\mathbb{N}}$ with $t_u < t_{u+1}$, for each u.

The investor bases her/his evaluation on the uniperiodal returns of the optimal portfolio computed in the ℓ dates before the trading day, where $\ell = 1, 2, \ldots, \bar{\ell}$ and $\bar{\ell}$ is the number of available past realizations of all the stocks returns. Obviously, the higher the value of ℓ, the wider the information set taken into account by the trading rule.

We introduce a time-dependent quantity $\mathcal{R}(t_u)$ which combines appropriately the information contained in the returns $\Pi^\star(t_u - t_{u-1}), \Pi^\star(t_{u-1} - t_{u-2}), \ldots, \Pi^\star(t_{u-\ell+1} - t_{u-\ell})$. As we will see, $\mathcal{R}(t_u)$ constitutes the base for the construction of the short-term trading rule. We also define a time-dependent *trading rule threshold* as follows:

$$\Lambda : \{t_u\}_{u\in\mathbb{N}} \to \mathbb{R} \qquad | \qquad t_u \mapsto \Lambda(t_u). \tag{18}$$

Function Λ reasonably depends on the riskless bond. Indeed, in a short-term perspective, investors must decide whenever to invest some shares of capital on the available risky assets or the entire value

in the riskless bond.
The trading algorithm runs as follows:

1a. set $u = \ell$;

2a. if $\mathcal{R}(t_u) \leq \Lambda(t_u)$, then trade on $\chi^\star(t_u)$ and go to step 3a.. Otherwise, do not trade on risky assets (invest the entire capital in the riskless bond, i.e. $\theta(t_u) = 0$, $\zeta(t_u) = 0$ and $\rho(t_u) = 1$) and go to step 3a.;

3a. set $t = t + 1$, and go to step 2a..

5 Monte Carlo experiments

The aim of this section is to provide, via numerical analysis, financial insights on the long-run strategy and on the short-term trading rule described in Sections 3 and 4, respectively. For this purpose, we explore the relationship between optimal long-term portfolio and short-term wealth allocation performances in the market considered, which is characterized by the presence of rarely traded stocks. At this aim, some Monte Carlo experiments are provided.
This section includes an analysis on several aspects of the model. In particular, we explore the sensitiveness of the results on the basis of different levels of frequency of trading in thin stocks, level of risk aversion and degree of correlation between risky assets. We do not carry out an analysis on the initial portfolio value: indeed, when comparing long- and short-run strategies at the end of the trading period, the role of this value is rather negligible.

5.1 Long-run optimal portfolio and short-term trading strategy

The general framework adopted in the numerical application can be synthesized as follows:

- Investors allocate their funds in a liquid and continuously traded risky asset with expected rate of return $\mu_1 = 0.05$ and volatility $\sigma_1 = 0.1$, in a light stock with expected rate of return $\mu_2 = 0.30$ and volatility $\sigma_2 = 0.60$, and in a riskless bond with constant rate of return, $r = 0.015$.
- The investment horizon in the long-term strategy is 3 years. So $T = 3 \times 240$ *business days*, i.e. $= 720$ *business days*. t_u measures *days* on yearly base, i.e. $t_u = u/240$, for each $u = 0, 1, \ldots, T$.
- The intraday trading is not allowed, so that in a business day it is assumed each stock is traded only at the opening of the market.

- The coefficient of risk aversion, γ, is alternatively set to 0.5 and 1.5.
- The frequency of jumps in thin stock price depends on the Poisson law parameter λ, which is alternatively set to 15 and 60.
- The correlation coefficient between the ordinary 1-dimensional Brownian motions B and W, namely ρ_{12}, is set alternatively to -0.5, 0 and 0.5.

The first two steps of the numerical procedure provide the simulation of the dynamics for $\{S_t\}_{t\geq 0}$ and $\{H_t\}_{t\geq 0}$ that are obtained replicating $K = 10,000$ times. Details on the general iterative method are reported in the Appendix.

In Figure 1 an example of a path for the price of a thin stock is reported for the case $\lambda = 15$.

[Insert Figure 1 about here]

Then we search for the optimal vector $\chi^\star$ of shares of capital to be invested in the three assets. At this aim, we perform a complete enumeration of the elements of a suitable admissible region $\mathcal{A}$ containing the portfolio paths in order to find the one maximizing the objective function J in (14). We test $M = 10,000$ portfolio paths of length T, where the values for each time t_u of the two shares $\theta(t_u)$ and $\zeta(t_u)$ have been extracted from a uniform distribution with support $[-1,1]$, while $\rho(t_u)$ is obtained by applying constraint (7). Starting from the K simulations of the prices of risky assets, for each portfolio m, with $m = 1,...M$, we construct the empirical time-dependent distribution of returns and compute the time-dependent mean and variance of the return of each portfolio, so that we can compute the optimal portfolio m^* as:

$$J_{m^\star} = \max_{m=1,\dots,M} J_m;$$

and the return of the optimal long-term portfolio:

$$\Pi^\star = \{\Pi^\star(t_{u+1} - t_u)\}_{u=1,\dots,T-1}.$$

More details are reported in the Appendix.

In accord with standard financial theory, the trading rule is based on the Sharpe ratios of the optimal portfolios found in the long-run strategy problem.

We fix a generic integer $\ell = 1,\dots,T$ and take into account the ℓ portfolio performances before the trading date. The trading rule is assumed to be based on the aggregation of the performance of the last ℓ periods. The quantity $R(t_u)$ is defined as follows:

$$\mathcal{R}(t_u) = \sum_{l=1}^{\ell} \frac{\mathbb{E}_x[\Pi^\star(t_{u-l+1} - t_{u-l})] - r}{\mathbb{V}_x[\Pi^\star(t_{u-l})]}. \tag{19}$$

The trading rule threshold is taken as a function of the riskless bond ℓ-periods accumulation interest rate:

$$\Lambda_u = e^{r(t_\ell - t_u)}, \qquad \forall\, u = 1, \dots, T. \tag{20}$$

In Monte Carlo experiments, we set $\ell = 2$. The trading rule allows to determine a short-term portfolio with time-dependent uniperiodal return $\Pi_S(t_{u+1} - t_u)$, which is selected and compared with the expected return of the long-run optimal wealth allocation strategy. The procedure is reported in the Appendix.

5.2 Results of the application and concluding remarks

As mentioned above, the main aim of the Monte Carlo experiments is to explore the relationship between long- and short-term trading strategies. The analysis is carried out by taking into consideration the influence of some variables on portfolio performances. In particular, the sensitiveness of the results on the basis of different levels of frequency trading, λ, risk aversion, γ, and degree of correlation between risky assets, ρ, is considered. In the Figures, the trajectories of the two returns related with the long- and short-term trading strategies for $\lambda = 15$ and $\lambda = 60$ are reported. In particular, Figure 2 shows these paths by setting the values for both risk aversion and correlation coefficients to $\gamma = 0.5$ and $\rho = -0.5$, respectively.

[Insert Figure 2 about here]

It is interesting to notice that in both the analyzed cases, at the achievement of the time-horizon, the long-term strategy appears to be preferable to that of short-term. In particular, the former offers, when compared to the latter, an excess return almost equal to 4%. This performance deteriorates slightly with increasing frequency of trading in the thin stock, and the excess returns reduces to 3%. In any case, it is interesting to note that in the $\lambda = 60$ case the long-term strategy performs always better than that of short-term. In the $\lambda = 15$ case, a certain alternation of position is observed in the convenience of a strategy relative to the other at the beginning of the time period, but gradually a predominance of the long-term strategy starts around the middle of the period considered.

Moving away from the case of negatively correlated returns, $\rho = -0.5$, and considering scenarios characterized by positive correlation between the returns of the risky assets, $\rho = 0.5$, in case of

$\gamma = 0.5$ the results are reversed and show an evident dominance of the short-term strategy, which concludes with a final return of the 7% against the negative final return of the long-term strategy equal to -9% (case $\lambda = 15$.) This finding is confirmed also in case of $\lambda = 60$.

[Insert Figure 3 about here]

As it is showed in Figure 4, results of positive correlation are confirmed also in case of risk aversion parameter $\gamma = 1.5$. When $\lambda = 15$, the short-term strategy clearly outperforms the long-term one and the excess return is almost equal to 5%. It is weird the evidence collected in the case of larger frequency in the trading of the thin stock, $\lambda = 60$, where it is observed that the final value of the portfolio in both the cases is almost equal, even though -during the investment period- the short-term strategy provides on average better results than the long-term one.

[Insert Figure 4 about here]

The case of uncorrelated risky assets offer a view on the relationship between long- and short-term strategies which confirms the outcome of the analysis in the case of positively correlated assets.
To conclude, we remark that the long-term strategy outperforms the short-term one only in case of negatively correlated returns between risky assets. In all the other cases, the optimal long-run portfolio exhibits negative returns at the final time T. An explanation of this finding is needed: the riskier the market position, the better taking financial decisions on the basis of a short horizons (see Figures 3 and 4). Given the parameters assumed in the numerical application, results of portfolio selection are in line with the diversification principle introduced in portfolio theory by Markowitz also when thin stocks are considered.

6 Appendix

In this Appendix we provide details and pseudo-codes of the numerical procedure described in Section 5
The first step of the experiment is represented by the simulation of the random times of the thin stock. The general iterative method for simulating random times consists in the following steps:

1b. set the starting time $\tau_0 = 0$;

2b. set $s = 0$;

3b. generate a random variable R_{s+1} from the exponential distribution with mean $1/\lambda$;

4b. set $\tau_{s+1} = \tau_s + R_{s+1}$;

5b. if $\tau_{s+1} > T$, stop. We denote the last index s as $\bar{s}$, i.e. $\tau_{\bar{s}} \leq T$ and $\tau_{\bar{s}+1} > T$. Otherwise, go to step 6b.;

6b. set $s = s + 1$ and go step 3b..

The procedure described above is replicated $K = 10,000$ times, so that the value of $\bar{s}$ obviously depends on the replication index, $k = 1, \ldots, K$. We denote as $\bar{s}(k)$ the value of the last index $\bar{s}$ for the k^{th} replication, and allocate random times in an incomplete matrix $\Upsilon = (\tau_s(k))_{s=1,\ldots,\bar{s}(k);\, k=1,\ldots,K}$.
To complete the matrix of random times a two steps procedure is needed. First, we insert the random times in the set of the trading days $\{t_1, \ldots, t_T\}$. In particular, we consider $\lceil \tau_s(k) \rceil / T$ where $\lceil \diamond \rceil$ indicates the superior integer part of $\diamond$. This choice is suitable in that each asset, by assumption, is traded at the beginning of a business day, which means the day after the jump occurrence. For the same reason, we assume that $\tau_s(k) = \tau_{s+1}(k)$ whenever $t_u < \tau_s(k) < \tau_{s+1}(k) < t_{u+1}$, for some $u \in \{0, 1, \ldots, T\}$. Second, we introduce the function $f^{(k)} : \{1, \ldots, T\} \to \{0, 1, \ldots, T\}$, such that:

$$f^{(k)}(t_u) = \begin{cases} t_u & \text{if } \exists\, s : t_u = \lceil \tau_s(k) \rceil / T; \\ \\ 0 & \text{otherwise.} \end{cases}$$

We denote such a complete matrix as $\mathcal{T}$, i.e.: $\mathcal{T} = \left(f^{(k)}(t_u)\right)_{u=1,\ldots,T;\, k=1,\ldots,K}$.
The second step of the numerical procedure consists in simulating the dynamics $\{S_t\}_{t\geq 0}$ and $\{H_t\}_{t\geq 0}$ in accord to equations (2) and (3), respectively. In doing so, we introduce the assumed correlation structure of the two 1-dimensional Brownian motions B and W and find the Cholesky matrix, namely C, for which $CC^T = \Sigma$.
The algorithm runs as follows:

1c. set $S_{t_0} = H_{t_0} = 10$;

2c. set $u = 0$;

3c. generate two independent random variables, $B_{t_{u+1}} \sim N(0, 1)$ and $W_{t_{u+1}} \sim N(0, 1)$;

4c. set

$$S_{t_{u+1}} = S_{t_u} \left(1 + \mu_1 (t_{u+1} - t_u) + \sqrt{t_{u+1} - t_u} C W_{t_{u+1}}\right)$$

and

$$H_{t_{u+1}} = H_{t_u} \left(1 + \mu_2 (t_{u+1} - t_u) + \sqrt{t_{u+1} - t_u} C B_{t_{u+1}}\right);$$

5c. set $u = u + 1$;

6c. if $u < T$, go to step 3c.; otherwise, stop.

The procedure described above is replicated $K = 10,000$ times. We identify the replication with a superscript $k = 1, \dots, K$, when needed.
The price of the thin stock is driven by $\{H_t\}_{t \geq 0}$ (see formula (3)) and jumps in accord to random times τ's. In order to derive the price series of the light stock we impose a control structure as follows:

- for each replication $k = 1, \dots, K$ and for each $u = 1, \dots, T$, if $\mathcal{T}(u, k) = 0$ then $H_{t_u}^{(k)} = H_{t_{u-1}}^{(k)}$, otherwise $H_{t_u}^{(k)} = H_{t_u}^{(k)}$.

Then, for each portfolio m, with $m = 1, \dots, M$.

1d. set $X_{t_0}^{(k)} = 10$, for each $k = 1, \dots, K$;

2d. set $m = 1$;

3d. set $u = 1$;

4d. generate two independent random variables $\theta(u)$ and $\zeta(u)$ from the uniform distribution with support $[-1, 1]$, and set $\rho(u) = 1 - \theta(u) - \zeta(u)$;

5d. starting from 4c., construct the empirical distribution of the value of the portfolio

$$X(t_u) = \left\{X_{t_u}^{(k)}\right\}_{k=1,\dots,K} = \left\{\theta(u)S_{t_u}^{(k)} + \zeta(u)H_{t_u}^{(k)} + (1 - \theta(u) - \zeta(u))B_{t_u}\right\}_{k=1,\dots,K};$$

6d. construct the empirical distribution of the uniperiodal returns of the portfolio as:

$$\Pi(t_u - t_{u-1}) = \{\Pi_k(t_u)\}_{k=1,\dots,K} = \left\{\frac{X_{t_u}^{(k)} - X_{t_{u-1}}^{(k)}}{X_{t_{u-1}}^{(k)}}\right\}_{k=1,\dots,K};$$

7d. set $u = u + 1$;

8d. if $u < T$, go to step 4d.; otherwise, go to step 9d.;

9d. calculate the objective function

$$J_m = \frac{1}{T} \cdot \sum_{u=0}^{T-1} e^{-\delta t_{u+1}} \left\{\mathbb{E}\left[\Pi(t_{u+1} - t_u)\right] - \gamma \mathbb{V}\left[\Pi(t_{u+1} - t_u)\right]\right\};$$

10d. set $m = m + 1$;

11d. if $m < M$, go to step 3d.; otherwise, go to step 12d.;

12d. compute $m^\star$ such that

$$J_{m^\star} = \max_{m=1,\dots,M} J_m.$$

Then, compute the return of the optimal portfolio as:
the return of the optimal long-term portfolio:

$$\Pi^\star = \{\Pi^\star(t_{u+1} - t_u)\}_{u=1,\dots,T-1}.$$

The procedure for computing the return of the short term trading strategy works as follows:

1e. set

$$\Pi_S(t_2 - t_1) = \mathbb{E}_x\left[\Pi^\star(t_2 - t_1)\right];$$

2e. set $u = \ell$;

3e. compute $\mathcal{R}(t_u)$;

4e. if $\mathcal{R}(t_u) \leq \Lambda_u$, then

$$\Pi_S(t_{u+1} - t_u) = \Pi_S(t_u - t_{u-1}) \cdot e^{r(t_{u+1}-t_u)}$$

and go to step 6e.;

5e. if $\mathcal{R}(t_u) > \Lambda_u$, then

$$\Pi_S(t_{u+1} - t_u) = \Pi_S(t_u - t_{u-1}) \cdot \mathbb{E}_x\left[\Pi^\star(t_{u+1} - t_u)\right]$$

and go to step 6e.;

6e. set $u = u + 1$;

7e. if $u < T$, go to step 3e.; otherwise, stop.

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Figure 1: Thin stock price dynamics, λ=15

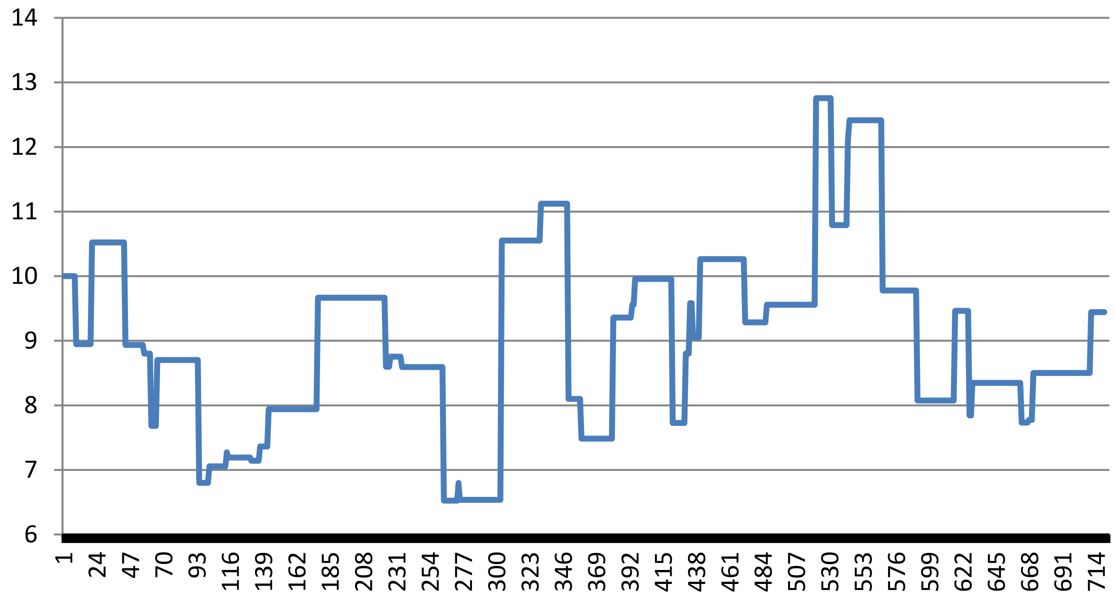

Figure 2: Long versus short term strategy, γ=0.5, ρ=-0.5

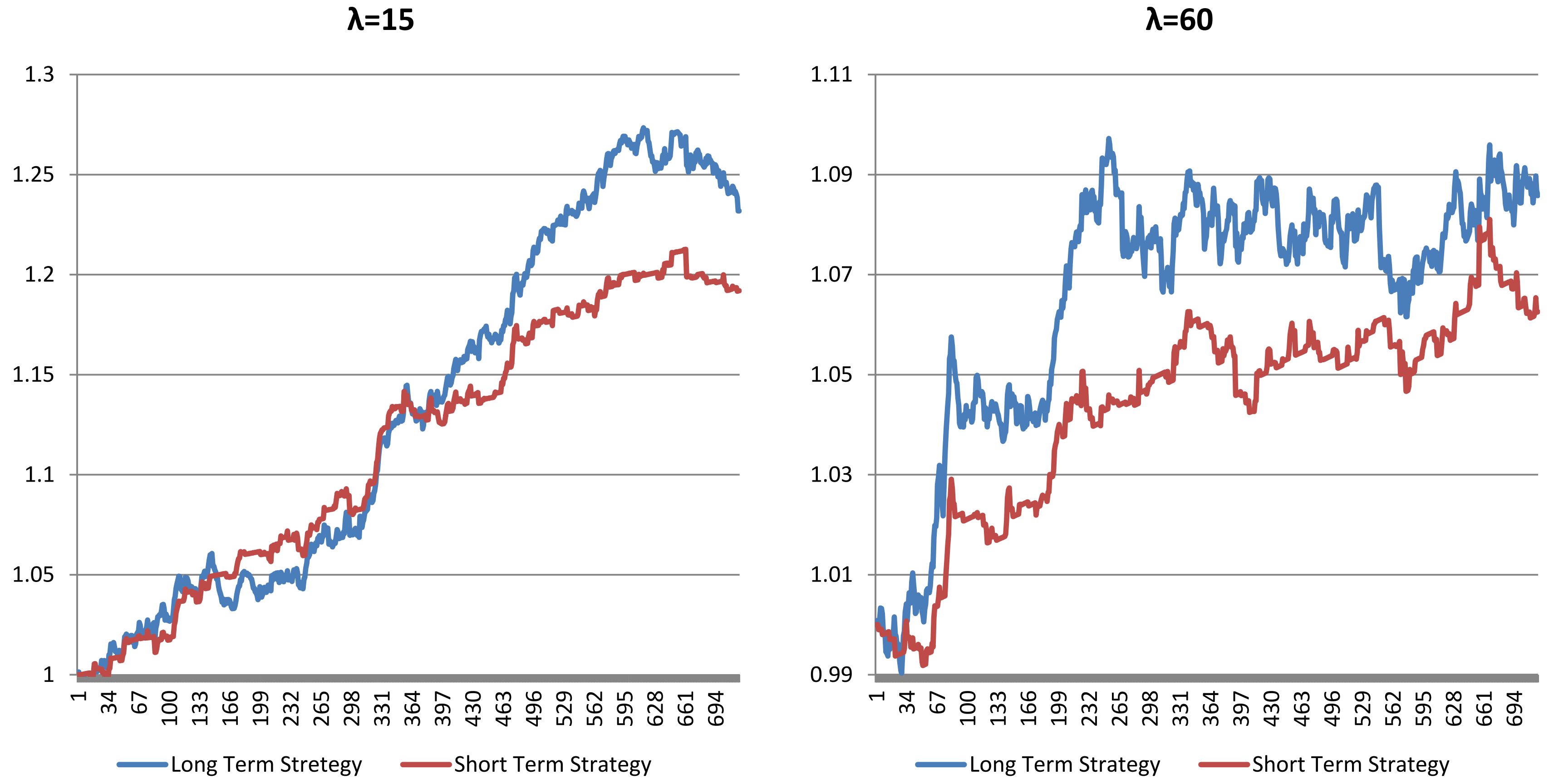

Figure 3: Long versus short term strategy, γ=0.5, ρ=0.5

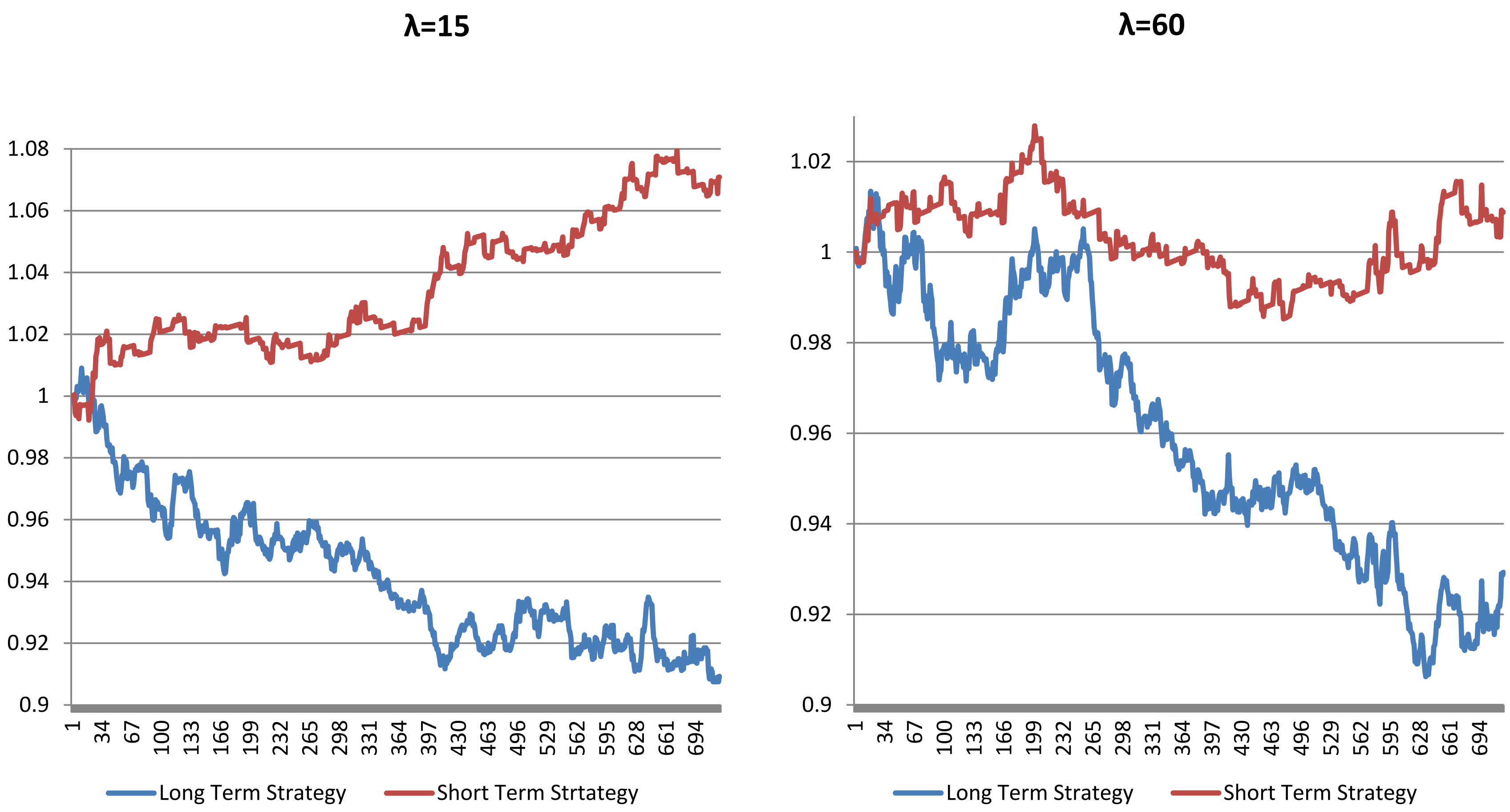

Figure 4: Long versus short term strategy, γ=1.5, ρ=0.5