Curves of Stationary Acceleration in $SE(3)$

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Abstract

The concept of curves of minimal acceleration seems to have been introduced by Žefran and Kumar and independently by Noakes, Heinzinger and Paden. In part the motivation was to extend the notion of spline curves to curves in groups, specifically the groups associated with robotics. A curve in the rigid body motion group $SE(3)$ for example, can be thought of as a trajectory of a rigid body. Hence these ideas have applications to motion planning and interpolation. In this work the analysis is repeated but using bi-invariant metrics on the group. Since these metrics are not positive definite the curves specified by the equations derived are only stationary, not minimal. It is possible to solve these non-linear coupled differential equations in some simple cases. However, these simple cases turn out to be highly relevant to robotics and mechanism theory.

1 Introduction

Understanding the geometry of rigid body motion is a fundamental problem in Robotics. A continuous sequence of rigid body motions can be thought of as a curve in the space of all possible rigid body motions, that is a curve in the group $SE(3)$. In robotics it is usual to assume that the links of the robot are rigid bodies. Hence, the motion of the robot's end-effector can be thought of as a curve in $SE(3)$.

A detailed understanding of the geometry of these curves would have many applications in robotics. For example a natural question to ask is: Given a starting position and a final target position for the end-effector of a robot, what is the 'best' path for the end-effector?

An early suggestion was that the robot should drive the end-effector along a screw motion. That is a rotation about a fixed axis followed by a translation in the direction of the same axis. There is almost always a unique screw motion which moves one position into the other. These motions correspond to one-parameter subgroups in the group of rigid body motions and are also geodesics in the group. A geodesic here is a curve with stationary arc-length with respect to a bi-invariant metric defined on the group. But currently these screw motions are not used. This is because some pairs of start and finish positions produce screw motions which produce excessively large motions of the end-effector.

This motion interpolation problem is also relevant in computer graphics where intermediate positions of bodies need to be generated between 'keyframes'. Many algorithms to generate spline curves have been proposed over the years both for computer graphics and robotics, however the geometry behind these methods is not too clear.

More recently work in biomechanics has suggested that humans move their limbs in such a way that jerk is minimised. However, there is a difficulty with this idea, jerk is the third derivative of displacement and in these biomechanical studies the jerk of some point on the hand is measured. It is not clear that the jerk of other points on the hand are also minimised. In part to generalise these idea to rigid body motions Žefran and Kumar (1998), defined the acceleration of a rigid body as the covariant derivative of its motion and the jerk as the second covariant derivative. Žefran and Kumar then studied curves which minimised these measures. By introducing the covariant derivative the curves obtained were automatically coordinate-free, that is invariant with respect to changes in coordinates or selection of reference point. This is clearly a desireable feature of any motion planning scheme, we don't want the path of the robot to depend on our choice of coordinate frame. Unfortunately, this work introduced another ambiguity, the covariant derivative depends on a choice of metric on the group. In order to minimise acceleration or jerk it is necessary that the metric should be positive definite.

But there are many such metrics to choose from on $SE(3)$. A little earlier Noakes *et al* (1989), used the same definition of acceleration to derive equations for minimum acceleration curves in $SO(3)$. They avoided the ambiguity refered to above by using the unique positive definite bi-invariant metric on the rotation group $SO(3)$. However, it is well known that there are no positive definite bi-invariant metrics on $SE(3)$. So instead Žefran and Kumar used positive definite left-invariant metrics. These left-invariant metrics can be thought of as inertia tensors for rigid bodies. Indeed the geodesics for such metrics, the minimum velocity curves, are simply the solutions to the dynamic equations for the rigid body not subject to any external forces. With this interpretation the main metric chosen by Žefran and Kumar is then the inertia tensor of a spherically symmetric body.

In this work the ideas of Žefran and Kumar are revisited but using bi-invariant metrics on $SE(3)$. Since these metrics are not positive definite the curves defined here are not going to be minimal—only stationary. In fact, neither Žefran and Kumar nor Noakes *et al* check that their curves are really minimal and the porperty of minimality is not subsequently used. The derivation presented here follows Žefran and Kumar but, not surprisingly, the results are essentially the same as Noakes *et al*, allowing for the change of group. However, since bi-invariant metrics are being studied here, standard results on the connection and its curvature can be used to shorten the exposition.

In fact the curves defined here do not depend on the precise bi-invariant metric used. So these curves really are intrinsic properties of the group, they do not depend on choice of coordinate frame or reference point. Hence, we can consider these curves as natural in some sense, and we can expect that they will be simple to deal with—at least for theoretical purposes. Although the equations for the curves are not in general soluble in closed form, there are there are many closed form solutions which can be found quite simply.

2 Screw Theory

In this section some of the mathematical background is given and basic notation defined.

A screw or more accurately a twist is an element of the Lie algebra to the group of rigid body motions $SE(3)$. In general the elements of a Lie algebra can be thought of a tangent vectors to the identity element in a group or equivalently as left-invariant vector fields on the group.

Screws can be written as 6-dimensional vectors, often partitioned into a pair of 3-vectors,

$$\mathbf{s} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{u} \end{pmatrix}, \tag{1}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the body and $\mathbf{u}$ a linear velocity characteristic of the motion. Corresponding to different representations of the group we can also have different representations of the Lie algebra. For example, we can write a screw as a 4×4 matrix,

$$S = \begin{pmatrix} \Omega & \mathbf{u} \\ 0 & 0 \end{pmatrix}, \tag{2}$$

again this is in partitioned form with Ω a 3×3 matrix. The relationship between the elements of $\boldsymbol{\omega}$ and those of Ω is given by,

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}. \tag{3}$$

The other representation of screws that used here is called the adjoint representation, this is a 6-dimensional representation defined by,

$$\mathrm{ad}(\mathbf{s}) = \begin{pmatrix} \Omega & 0 \\ U & \Omega \end{pmatrix}. \tag{4}$$

Here U is the 3×3 anti-symmetric matrix corresponding to $\mathbf{u}$ in the same way that Ω corresponds to $\boldsymbol{\omega}$.

In any Lie algebra we have a binary operation called the Lie bracket or commutator. The operation is denoted, $[\mathbf{s}_1,\, \mathbf{s}_2]$ and in a matrix representation it is given by the commutator of matrices,

$$\mathrm{ad}([\mathbf{s}_1,\, \mathbf{s}_2]) = \mathrm{ad}(\mathbf{s}_1)\,\mathrm{ad}(\mathbf{s}_2) - \mathrm{ad}(\mathbf{s}_2)\,\mathrm{ad}(\mathbf{s}_1). \tag{5}$$

The Lie bracket is anti-commutative,

$$[\mathbf{s}_1,\, \mathbf{s}_2] = -[\mathbf{s}_2,\, \mathbf{s}_1]. \tag{6}$$

It is not associative but it does satisfy the Jacobi identity:

$$[\mathbf{s}_1,\ [\mathbf{s}_2, \mathbf{s}_3]] + [\mathbf{s}_2,\ [\mathbf{s}_3, \mathbf{s}_1]] + [\mathbf{s}_3,\ [\mathbf{s}_1, \mathbf{s}_2]] = 0. \tag{7}$$

The adjoint representation has the property that for any Lie algebra elements,

$$\mathrm{ad}(\mathbf{s}_1)\mathbf{s}_2 = [\mathbf{s}_1,\ \mathbf{s}_2]. \tag{8}$$

The Lie algebra can be mapped to the group using the exponential map,

$$e^S = I + S + \frac{1}{2}S^2 + \frac{1}{3!}S^3 + \cdots. \tag{9}$$

If S is in some matrix representation of the Lie algebra then e^S will be in the corresponding representation of the group. In particular, if we exponentiate an matrix from the adjoint representation of the Lie algebra we will get a matrix in the adjoint representation of the group, denoted $\mathrm{Ad}(G)$ here.

Since square matrices always satisfy a polynomial equation (Cayley-Hamilton theorem) the infinite sum of matrix powers in the definition of the exponential map above, is a little misleading. In fact we can usually write the exponential map in terms of just the first few powers of the matrix. However, the precise expression may depend on the representation chosen rather than the Lie algebra element itself. In $SE(3)$ we have simple results that will be used later. If $\mathbf{s}$ is a pure translation, that is if $\boldsymbol{\omega} = \mathbf{0}$, then we have the particularly simple result,

$$e^S = I + S. \tag{10}$$

On the other hand if $\mathbf{s}$ is a unit rotation about the origin, that is $\mathbf{u} = \mathbf{0}$ and $\boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1$, then an arbitrary rotation about the origin can be written $\theta\mathbf{s}$ where θ is the rotation angle. Now the exponential of this is,

$$e^{\theta S} = I + \sin\theta S + (1 - \cos\theta)S^2. \tag{11}$$

This is the well known Rodrigues formula. In this case the formulas are the same for both the 4-dimension representation and the adjoint representation. It is possible to derive formulas for more general elements of the Lie algebra but these will not be needed in what follows.

Suppose that we have curve in a Lie group given by the exponential of a curve in the Lie algebra,

$$G(t) = e^{S(t)}, \tag{12}$$

where t is the parameter along the curve. Now, what is the derivative of such a curve? If the curve in the Lie algebra is simply some scalar function of t times a constant screw then we have,

$$\frac{d}{dt}e^{\theta(t)S} = \dot{\theta}(t)e^{\theta(t)S}. \tag{13}$$

However, when the screw itself is a function of t things are not so simple. This is because the matrices $\dot{S}$ and S do not necessarily commute. Hausdorff (1906) showed, that in general we have

$$\frac{d}{dt}e^{S(t)} = S_d e^{\theta(t)S}, \tag{14}$$

where

$$S_d = \dot{S} + \frac{1}{2}[S,\, \dot{S}] + \frac{1}{3!}[S,\, [S, \dot{S}]] + \frac{1}{4!}[S, [S, [S, \dot{S}]]] + \cdots. \tag{15}$$

This can be written more neatly in the adjoint representation as

$$\mathrm{ad}(\mathbf{s}_d) = \sum_{i=0}^{\infty} \frac{1}{(i+1)!}\,\mathrm{ad}(\mathbf{s})^i \dot{\mathbf{s}}. \tag{16}$$

Again it is possible to find reasonably short formulas for S_d however these will not be required below. More important here is the interpretation of this result. The derivative is the tangent vector to the curve. Translating the tangent vector back to the identity gives the corresponding Lie algebra element,

$$\frac{dG(t)}{dt}G^{-1}(t) = S_d. \tag{17}$$

On $SE(3)$ we have two bi-invariant metrics, or rather a pencil of them. We will write any of these metrics as,

$$< \mathbf{s}_1,\, \mathbf{s}_2 >= \mathbf{s}_1^T Q_p \mathbf{s}_2. \tag{18}$$

Here Q_p is a 6×6 matrix,

$$Q_p = \begin{pmatrix} -2\alpha I_3 & \beta I_3 \\ \beta I_3 & 0 \end{pmatrix}, \tag{19}$$

with $p = \alpha/\beta$ and I_3 the 3×3 identity matrix. When $p = 0$ we have,

$$Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}. \tag{20}$$

This is sometimes call the Klein form or the reciprocal form. If $\beta = 0$ but $\alpha \neq 0$ we set $p = \infty$ and

$$Q_\infty = \begin{pmatrix} -2I_3 & 0 \\ 0 & 0 \end{pmatrix}. \tag{21}$$

This is the Killing form of the Lie algebra. Notice that Q_∞ is semi-definite and all the other metrics are indefinite. However, Q_∞ is also the only one of these metrics that is degenerate. The group invariance comes from the fact that,

$$\mathrm{Ad}^T(G) Q_p Ad(G) = Q_p, \tag{22}$$

for all p and all group elements $G \in SE(3)$, hence

$$< \mathrm{Ad}(G)\mathbf{s}_1,\, \mathrm{Ad}(G)\mathbf{s}_2 > = < \mathbf{s}_1,\, \mathbf{s}_2 > . \tag{23}$$

The covariant derivative is a differential operator on vector fields. The definition of the covariant derivative depends on a choice of metric. We also usually demand that the covariant derivative be torsion free, this then determines a unique covariant derivative. In a Lie group the covariant derivative based on a bi-invariant metric (provided such a metric exists) satisfies,

$$\nabla_X Y = \frac{1}{2}[X,\, Y], \tag{24}$$

for arbitrary left-invariant vector fields X and Y, that is for Lie algebra elements X and Y. Further details of these covariant derivatives will be introduced later. More details on rigid body motions, screws and covariant derivatives can be found in (Selig 2005) and similar texts.

3 Acceleration

Recently in the robotics literature there has been some discussion about acceleration and second derivatives. See (Featherstone 2001), (Stramigiolli and Bruyninckx 2001) and also to some extent (Lipkin 2005). This interest seems to date back to a discussion at a conference in Cambridge a few years ago. The difficulty is that, “Acceleration is not a screw!”. This statement makes perfect sense if one considers the motion of points in space to be the primary object of study. Consider an arbitrary point $\mathbf{r}$, on a rigid body moving according to some sequence of rigid transformations $G(t) \in SE(3)$. Then the point on the rigid body, which has position $\mathbf{r}_0$ at time $t = 0$ will subsequently have position given by,

$$\begin{pmatrix} \mathbf{r}(t) \\ 1 \end{pmatrix} = G(t) \begin{pmatrix} \mathbf{r}_0 \\ 1 \end{pmatrix}. \tag{25}$$

Now the 4×4 matrix $G(t)$ can be written as the exponential of a twist $S(t)$,

$$G(t) = e^{S(t)}, \tag{26}$$

where $S(t)$ lies in the 4×4 representation of the Lie algebra. The velocity of the point is now easy to compute by differentiating,

$$\begin{pmatrix} \dot{\mathbf{r}}(t) \\ 0 \end{pmatrix} = S_d e^{S(t)}, \qquad \begin{pmatrix} \mathbf{r}_0 \\ 1 \end{pmatrix} = S_d \begin{pmatrix} \mathbf{r}(t) \\ 1 \end{pmatrix}, \tag{27}$$

where S_d is the “velocity screw” of the rigid body. Notice that if we expand this relation using the partitioned form the following relation can be derived in terms of 3-dimensional vectors,

$$\dot{\mathbf{r}} = \boldsymbol{\omega}_d \times \mathbf{r} + \mathbf{v}_d. \tag{28}$$

This is the standard form for the velocity field given by the point in a rigid body moving about an instantaneous screw S_d.

The acceleration field of these points is not given by such a simple relation. However, we can find the relation for the acceleration of points if we just differentiate again,

$$\begin{pmatrix} \ddot{\mathbf{r}}(t) \\ 0 \end{pmatrix} = \dot{S}_d(t) e^{S(t)} \begin{pmatrix} \mathbf{r}_0 \\ 1 \end{pmatrix} + \big(S_d(t)\big)^2 e^{S(t)} \begin{pmatrix} \mathbf{r}_0 \\ 1 \end{pmatrix}. \tag{29}$$

So that,

$$\begin{pmatrix} \ddot{\mathbf{r}}(t) \\ 0 \end{pmatrix} = \Big(\dot{S}_d(t) + \big(S_d(t)\big)^2\Big) \begin{pmatrix} \mathbf{r}(t) \\ 1 \end{pmatrix}. \tag{30}$$

Expanding this we get the result,

$$\ddot{\mathbf{r}} = \dot{\boldsymbol{\omega}}_d \times \mathbf{r} + \dot{\mathbf{v}}_d + \boldsymbol{\omega}_d \times (\boldsymbol{\omega}_d \times \mathbf{r}) + \boldsymbol{\omega}_d \times \mathbf{v}_d. \tag{31}$$

Now, since S_d is an element of a Lie algebra and hence a vector in a vector space, its derivative $\dot{S}_d$, is also an element of the same vector space. So we see that $\dot{S}_d$ is a screw but that the acceleration of a point is not simply $\dot{S}_d\mathbf{r}$, there is an extra term which involves the square of the velocity screw.

Žefran and Kumar defined the acceleration of a rigid body using a covariant derivative. Let us write the tangent vector to the curve in the group as $V = S_d$. Then in Žefran and Kumar's notation the acceleration is,

$$A = \nabla_V V. \tag{32}$$

Now we may identify left-invariant vector fields with Lie algebra elements, so we will choose a basis for the Lie algebra,

$$\boldsymbol{\omega}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\omega}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\omega}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathbf{v}_x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{v}_y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{v}_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We can write the tangent vector field to a curve as,

$$V = a_1(t)\boldsymbol{\omega}_x + a_2(t)\boldsymbol{\omega}_y + a_3(t)\boldsymbol{\omega}_z + a_4(t)\mathbf{v}_x + a_5(t)\mathbf{v}_y + a_6(t)\mathbf{v}_z, \tag{33}$$

where the coefficients a_i are functions of the position along the curve. In section 2 above we saw that for a covariant derivative compatible with bi-invariant metrics, acting on Lie algebra elements we have

$$\nabla_X Y = \frac{1}{2}[X, Y], \tag{34}$$

see (Milnor 1969) and §15.1 of (Selig 2005). Notice that this means that the basis chosen is not a coordinate basis since in general the Lie algebra elements do not commute.

To find an expression for the acceleration $\nabla_V V$, we may use the following standard identities for a covariant derivative taken from (Schutz 1980):

$$\begin{aligned} \nabla_X(aY) &= a\nabla_X Y + X\frac{da}{dt}, & (35) \\ \nabla_{aX+bY} Z &= a\nabla_X Z + b\nabla_Y Z. & (36) \end{aligned}$$

Using the general form for the tangent vector to a curve given in (33) above we get,

$$\nabla_V V = a_1 \nabla_V \boldsymbol{\omega}_x + a_2 \nabla_V \boldsymbol{\omega}_y + \cdots + a_6 \nabla_V \mathbf{v}_z + \boldsymbol{\omega}_x \frac{da_1}{dt} + \cdots + \mathbf{v}_z \frac{da_6}{dt}. \quad (37)$$

Now using the relations above we have for example,

$$\begin{aligned} \nabla_V \boldsymbol{\omega}_x &= a_1 \nabla_{\boldsymbol{\omega}_x} \boldsymbol{\omega}_x + \cdots + a_6 \nabla_{\mathbf{v}_z} \boldsymbol{\omega}_x, \\ &= \frac{1}{2} a_1 [\boldsymbol{\omega}_x, \boldsymbol{\omega}_x] + \cdots + \frac{1}{2} a_6 [\mathbf{v}_z, \boldsymbol{\omega}_x], \\ &= \frac{1}{2}[V, \boldsymbol{\omega}_x], & (38) \end{aligned}$$

and hence we have that

$$\nabla_V V = \frac{1}{2}[V,\, V] + \dot{V} = \dot{V}. \quad (39)$$

That is $\nabla_V V = \dot{S}_d$. This agrees with the results given by Žefran and Kumar, even though they used a different metric.

Finally here a similar expression for the jerk can be found. Jerk is roughly the third derivative of position and was defined by Žefran and Kumar as $\nabla_V \nabla_V V$. With respect to a bi-invariant metric we find that

$$\nabla_V \nabla_V V = \ddot{V} + \frac{1}{2}[V,\, \dot{V}]. \quad (40)$$

4 Stationary Acceleration

Next we look at the problem of optimising the acceleration along a curve. Again following Žefran and Kumar we define the acceleration along a curve by the integral,

$$J = (1/2) \int_a^b < \nabla_V V,\, \nabla_V V > \, dt. \quad (41)$$

Here however, the notation $< X, Y >$ denotes a bi-invariant metric, see section 2 above.

Following the derivation of Žefran and Kumar yet again, we define a first variation of the integral as a vector field S, which vanishes at the boundary points a and b. Further we require that the covariant derivatives of S along V; $\nabla_V S$, vanish at the boundary points. Now the Lie derivative of the integral J along the field S is given by,

$$\begin{aligned} \pounds_S J &= (1/2)\int_a^b S < \nabla_V V, \nabla_V V > dt \\ &= \int_a^b < \nabla_S \nabla_V V, \nabla_V V > dt \end{aligned} \tag{42}$$

We can swap the order of the covariant derivatives using the definition of the curvature of the metric R,

$$R(X, Y)W = -\nabla_X \nabla_Y W + \nabla_Y \nabla_X W + \nabla_{[X,Y]} W. \tag{43}$$

Notice that S and V do not commute, this is not a coordinate basis, see (Schutz 1980). For brevity we will consider the integrand $\mathcal{I} =< \nabla_S \nabla_V V, \nabla_V V >$, so that,

$$\mathcal{I} =< \nabla_V \nabla_S V + \nabla_{[S,V]} V - R(S, V)V, \nabla_V V > . \tag{44}$$

The first term here $< \nabla_V \nabla_S V, \nabla_V V >$ can be written as, $V < \nabla_S V, \nabla_V V > - < \nabla_S V, \nabla_V^2 V >$. The first of these terms is a total derivative and hence can be integrated. In general,

$$\nabla_X Y = \nabla_Y X + [X, Y], \tag{45}$$

so that, since we have assumed that S and $\nabla_V S$ vanish at the boundaries so does $\nabla_S V$ and hence so does the integral of this term.

The relation (45) expresses the fact that the connection we are using has no torsion. It can be used to develop the the rest of the first term:

$$< \nabla_S V, \nabla_V^2 V >=< \nabla_V S + [S, V], \nabla_V^2 V > . \tag{46}$$

Again we can remove the first derivative with respect to V,

$$< \nabla_S V, \nabla_V^2 V >= V < S, \nabla_V^2 V > - < S, \nabla_V^3 V > + < [S, V], \nabla_V^2 V > . \tag{47}$$

Once again the first term in the above equation can be integrated but then vanishes at the boundary points. The second term in the integrand $\mathcal{I}$, can be manipulated in a similar fashion to produce the overall result:

$$\pounds_S J = \int_a^b < S,\, \nabla_V^3 V > -2 < [S,V],\, \nabla_V^2 V > \\ + < [[S,V],\, V],\, \nabla_V V > - < R(S,V)V,\, \nabla_V V > \, dt. \quad (48)$$

Next we use a couple of relations which apply to bi-invariant metrics in Lie groups and hence were not available to Žefran and Kumar. First,

$$< [X,Y],\, Z > = < X,\, [Y,Z] > . \quad (49)$$

In the Lie algebra of the rotation group $so(3)$, the bi-invariant metric is the scalar product of 3-vectors and the Lie bracket is the vector product. So the above relation could be thought of as the generalisation of the cyclic property of the scalar triple product to arbitrary Lie algebras. The second relation we have is that the curvature is given by,

$$R(X,Y)Z = \frac{1}{4}[[X,Y],\, Z]. \quad (50)$$

These relations can be found in (Milnor 1969). Hence the first variation of the integral becomes,

$$\pounds_S J = \int_a^b < S,\, \nabla_V^3 V + 2[\nabla_V^2 V, V] + \frac{3}{4}[[\nabla_V V, V], V] > \, dt. \quad (51)$$

In order that this vanish for arbitrary variations S it is necessary that,

$$\nabla_V^3 V + 2[\nabla_V^2 V, V] + \frac{3}{4}[[\nabla_V V, V], V] = 0, \quad (52)$$

or, if we substitute $\nabla_V V = \dot{V}$ then the equation for stationary acceleration is

$$\nabla_V^2 \dot{V} + 2[\nabla_V \dot{V}, V] + \frac{3}{4}[[\dot{V}, V], V] = 0. \quad (53)$$

Recall from (40) above that,

$$\nabla_V \dot{V} = \frac{1}{2}[V,\, \dot{V}] + \ddot{V} \quad (54)$$

and hence

$$\nabla_V \nabla_V \dot{V} = \frac{1}{4}[V,\, [V,\, \dot{V}]] + [V,\, \ddot{V}] + V^{(3)}. \quad (55)$$

Substituting this into (53) above gives

$$V^{(3)} + [\ddot{V},\, V] = 0. \tag{56}$$

This agrees with the results of Noakes *et al* (1989) who derive a relation for stationary acceleration curves of the bi-invariant metric in $SO(3)$.

5 First integrals

Noakes *et al* (1989), observe that equation (56) can be integrated once to give

$$\ddot{V} + [\dot{V},\, V] = C, \tag{57}$$

where the constant vector C is determined by the boundary conditions. Solutions to this equation are called Lie quadratics by Noakes and when $C = 0$ they are null Lie quadratics, see(Noakes 2003).

In (Noakes 2003) it is also established that the scalar $\mathcal{J} =< \ddot{V}, \ddot{V} >$ is a constant along the solutions to (56) in $SO(3)$. This can be show very simply here for any group with a bi-invariant metric. To see this we differentiate this quantity along the path and show that its derivative vanishes. So consider,

$$\begin{aligned} V < \ddot{V},\, \ddot{V} > &= 2 < \ddot{V},\, \nabla_V \ddot{V} >, \\ &= 2 < \ddot{V},\, (V^{(3)} + [\ddot{V}, V]) - \frac{3}{2}[\ddot{V}, V] >= 0. \end{aligned} \tag{58}$$

The term in the round brackets here is the equation for stationary acceleration and hence vanishes along the curve, see equation (56) above. The second term vanishes because of the triple product identity (49) above.

There is however, an alternative expression for the first integral of (56). This is given by,

$$\ddot{V} = GXG^{-1}, \tag{59}$$

where X is a constant vector. To see this we can differentiate the above equation remembering that $dG/dt = VG$ see (17) above.

$$V^{(3)} = VGXG^{-1} + GX\frac{d}{dt}G^{-1}. \tag{60}$$

To find dG^{-1}/dt we can differentiate the relation $GG^{-1} = I$ to show that $dG^{-1}/dt = -G^{-1}V$. Hence,

$$V^{(3)} = VGXG^{-1} + GXG^{-1}V = [V, \ddot{V}]. \tag{61}$$

Notice that it is clear that the solution $\ddot{V} = GXG^{-1}$ has constant $\mathcal{J}$ indeed $< \ddot{V},\, \ddot{V} >=< X, X >$.

Finally here notice that we can play the same trick for null Lie quadratics. When $C = 0$, $\dot{V} = GYG^{-1}$ will satisfy (57) and it is easy to see that $< \dot{V}, \dot{V} >$ will be constant along these curves.

6 Simplest Solutions

A very simple and obvious solution to (53) occurs if we let $G = e^{\theta X}$ with θ a function of t. Then $V = \dot{\theta} X$ and hence $\ddot{V} = \theta^{(3)} X$. We obtain a solution so long as $\theta^{(3)} = 1$. That is, whenever θ is a cubic polynomial in t,

$$G(t) = e^{(t^3/6 + c_2 t^2 + c_1 t + c_0)X}. \tag{62}$$

This is a motion about a single screw axis but the “rate of screwing” is not constant but a cubic polynomial.

This solution can be elaborated a little, notice it was only really necessary that the Lie algebra exponent in G commute with the constant X. In the group $SO(3)$ the centraliser of any element, that is the set of elements which commute with it, are trivial. In $SE(3)$ this is not the case, for example all the translations commute with each other. So if $X = \mathrm{ad}(\mathbf{t})$, where $\mathbf{t}$ is a translation, then we have a simple solution,

$$G(t) = e^{\theta\, \mathrm{ad}(\mathbf{t}) + \phi_1\, \mathrm{ad}(\mathbf{t}_1) + \phi_2\, \mathrm{ad}(\mathbf{t}_2)}, \tag{63}$$

where θ is a cubic in t as above, ϕ_1 and ϕ_2 are quadratics in t, and $\mathbf{t}_1$ and $\mathbf{t}_2$ are translations linearly independent of $\mathbf{t}$.

For a general element of the Lie algebra, that is a screw motion with finite pitch we can always decompose the element into a rotation and a translation along the same axis,

$$X = \mathrm{ad}(\boldsymbol{\omega}) + \mathrm{ad}(\mathbf{t}). \tag{64}$$

Moreover, the rotation $\mathrm{ad}(\boldsymbol{\omega})$ and the translation $\mathrm{ad}(\mathbf{t})$ will commute since they have the same axis. Hence we have a simple solution,

$$G(t) = e^{\theta_1 \,\mathrm{ad}(\boldsymbol{\omega}) + \theta_2 \,\mathrm{ad}(\mathbf{t})} = e^{\theta_1 \,\mathrm{ad}(\boldsymbol{\omega})} e^{\theta_2 \,\mathrm{ad}(\mathbf{t})}, \tag{65}$$

where θ_1 and θ_2 are both cubic polynomials in t. Notice that this motion is quite similar to our first solution (62) above but now the pitch of the screw can also vary but the axis of the screw remains fixed.

Finally here, suppose that,

$$G(t) = e^{t\,\mathrm{ad}(\mathbf{v})} e^{t\,\mathrm{ad}(\boldsymbol{\omega})}, \tag{66}$$

where the axes of the rotation $\boldsymbol{\omega}$ and the translation $\mathbf{v}$ are arbitrary. For such a motion we have,

$$V = \frac{d}{dt} G\, G^{-1} = \mathrm{ad}(\mathbf{v}) + e^{t\,\mathrm{ad}(\mathbf{v})}\,\mathrm{ad}(\boldsymbol{\omega}) e^{-t\,\mathrm{ad}(\mathbf{v})}, \tag{67}$$

and hence,

$$\dot{V} = e^{t\,\mathrm{ad}(\mathbf{v})}[\mathrm{ad}(\mathbf{v}),\, \mathrm{ad}(\boldsymbol{\omega})] e^{-t\,\mathrm{ad}(\mathbf{v})}, \tag{68}$$

and subsequently,

$$\ddot{V} = e^{t\,\mathrm{ad}(\mathbf{v})}[\mathrm{ad}(\mathbf{v}),\, [\mathrm{ad}(\mathbf{v}),\, \mathrm{ad}(\boldsymbol{\omega})]] e^{-t\,\mathrm{ad}(\mathbf{v})}. \tag{69}$$

Now the commutator $[\mathrm{ad}(\mathbf{v}),\, \mathrm{ad}(\boldsymbol{\omega})]$, between a translation and a rotation is always another translation. So the double commutator $[\mathrm{ad}(\mathbf{v}),\, [\mathrm{ad}(\mathbf{v}),\, \mathrm{ad}(\boldsymbol{\omega})]]$, is a commutator between two translations and hence vanishes. Thus this motion satisfies (59) with the constant $X = 0$. Notice that the vanishing X in (59) is different from the null Lie quadratics discussed by Noakes (2003).

Notice that if the translation vector $\mathbf{t}$ and the rotation axis $\boldsymbol{\omega}$ are perpendicular then this is a planar motion, such a motion is illustrated in figure 1. The motion is illustrated by a sequence of positions for a pair of small perpendicular lines. These lines could be thought of as a coordinate frame in the moving body. Notice that the origin, where the lines meet traces out a cycloid. In the next section planar motions will be studied in more detail.

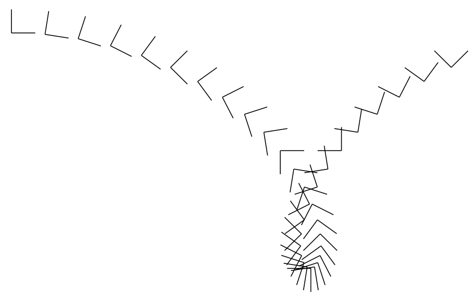

Figure 1: A Stationary Acceleration Curve in the Plane

7 Planar Motions

Another way to simplify the problem is to restrict to a subgroup of full rigid-body motion group. Here we look at the group of planar motions $SE(2)$. As generators of the Lie algebra of this group we can take the two translations,

$$\mathbf{s}_1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{i} \end{pmatrix}, \text{ and } \mathbf{s}_2 = \begin{pmatrix} \mathbf{0} \\ \mathbf{j} \end{pmatrix},$$

together with the rotation

$$\mathbf{s}_3 = \begin{pmatrix} \mathbf{k} \\ \mathbf{0} \end{pmatrix}.$$

Now we can write down the equations in terms of canonical coordinates of the second kind, see (Norman and Wei 1964). That is we seek a solution of the form,

$$G(t) = e^{\theta_1 \,\mathrm{ad}(\mathbf{s}_1)} e^{\theta_2 \,\mathrm{ad}(\mathbf{s}_2)} e^{\theta_3 \,\mathrm{ad}(\mathbf{s}_3)}, \tag{70}$$

where the joint variables θ_i are functions of t but the elements $\mathbf{s}_i$ are constant. With the generators as defined above the exponentials can be expressed as

$$e^{\theta_1 \,\mathrm{ad}(\mathbf{s}_1)} = I + \theta_1 \,\mathrm{ad}(\mathbf{s}_1), \tag{71}$$

$$e^{\theta_2 \,\mathrm{ad}(\mathbf{s}_2)} = I + \theta_2 \,\mathrm{ad}(\mathbf{s}_2), \tag{72}$$

and

$$e^{\theta_3 \,\mathrm{ad}(\mathbf{s}_3)} = I + \sin\theta_3 \,\mathrm{ad}(\mathbf{s}_3) + (1 - \cos\theta_3)\,\mathrm{ad}(\mathbf{s}_3)^2. \tag{73}$$

Moreover, $\mathrm{ad}(\mathbf{s}_1)$ commutes with $\mathrm{ad}(\mathbf{s}_2)$ and

$$[\mathrm{ad}(\mathbf{s}_1),\, \mathrm{ad}(\mathbf{s}_3)] = -\,\mathrm{ad}(\mathbf{s}_2),\ [\mathrm{ad}(\mathbf{s}_2),\, \mathrm{ad}(\mathbf{s}_3)] = \mathrm{ad}(\mathbf{s}_1). \tag{74}$$

Using these relations we find expressions for $V = \dot{G}G^{-1}$ and its derivatives,

$$\begin{aligned} V &= (\dot{\theta}_1 + \dot{\theta}_3\theta_2)\,\mathrm{ad}(\mathbf{s}_1) + (\dot{\theta}_2 - \dot{\theta}_3\theta_1)\,\mathrm{ad}(\mathbf{s}_2) + \dot{\theta}_3\,\mathrm{ad}(\mathbf{s}_3), \quad (75)\\ \dot{V} &= (\ddot{\theta}_1 + \dot{\theta}_3\dot{\theta}_2 + \ddot{\theta}_3\theta_2)\,\mathrm{ad}(\mathbf{s}_1) \\ &\quad +(\ddot{\theta}_2 - \dot{\theta}_3\dot{\theta}_1 - \ddot{\theta}_3\theta_1)\,\mathrm{ad}(\mathbf{s}_2) + \ddot{\theta}_3\,\mathrm{ad}(\mathbf{s}_3), \quad (76)\\ \ddot{V} &= (\theta_1^{(3)} + \dot{\theta}_3\ddot{\theta}_2 + 2\ddot{\theta}_3\dot{\theta}_2 + \theta_3^{(3)}\theta_2)\,\mathrm{ad}(\mathbf{s}_1) \\ &\quad +(\theta_2^{(3)} - \dot{\theta}_3\ddot{\theta}_1 - 2\ddot{\theta}_3\dot{\theta}_1 - \theta_3^{(3)}\theta_1)\,\mathrm{ad}(\mathbf{s}_2) + \theta_3^{(3)}\,\mathrm{ad}(\mathbf{s}_3). \quad (77) \end{aligned}$$

Now the constant vector X in (59) can be chosen to be,

$$X = r\cos\delta\,\mathrm{ad}(\mathbf{s}_1) + r\sin\delta\,\mathrm{ad}(\mathbf{s}_2) + \phi\,\mathrm{ad}(\mathbf{s}_3). \quad (78)$$

Substituting in to the equation $\ddot{V} = GXG^{-1}$ and comparing the coefficients of the generators gives three equations, after a little cancelation these are,

$$\begin{aligned} \theta^{(3)} &= \phi, \quad (79)\\ \theta_1^{(3)} + \dot{\theta}_3\ddot{\theta}_2 + 2\ddot{\theta}_3\dot{\theta}_2 &= r\cos(\theta_3 + \delta), \quad (80)\\ \theta_2^{(3)} - \dot{\theta}_3\ddot{\theta}_1 - 2\ddot{\theta}_3\dot{\theta}_1 &= r\sin(\theta_3 + \delta). \quad (81) \end{aligned}$$

Clearly the general solution for the rotation angle θ_3 from equation (79), is a cubic in t. If we substitute this in the equations (80) and (81) we get a pair of coupled linear equations for the translation variables θ_1 and θ_2. These equations are probably solvable in closed form but certainly a general solution is not easy to write down.

Again we will just look at a couple of simple solutions. First, assume that $X = \mathbf{0}$ that is $\phi = 0$ and $r = 0$. Now suppose we have $\theta_3 = \alpha t + \beta$ as a possible solution for the rotation, with α and β constant. Substituting this into equations (80) and (81) gives a pair of homogeneous, constant-coefficient linear equations. The general solution for the translational variables will be a sinusoid plus a term linear in t,

$$\begin{aligned} \theta_1 &= A\cos(\alpha t + \gamma) + Bt + C, \quad (82)\\ \theta_2 &= A\sin(\alpha t + \gamma) + Dt + E, \quad (83) \end{aligned}$$

where are A, B, C, D, E and γ are arbitrary constants. In the case were the constant $A = 0$ all the variables θ_i are linear functions of t. This is a motion much used in industrial robotics, it consists of a translation along a line with a simultaneous rotation about a perpendicular axis. Notice that this is not a screw motion, where the rotation and translation axes would be parallel. In fact these motions can be thought of as being generated by a circle rolling on a line, the radius of the circle will be determined by the constants α, B and D. Any point on the moving plane will trace out a cycloid. Of course, this is the motion mentioned at the end of the last section and illustrated in figure 1.

In the case where $B = D = 0$ but $A \neq 0$, the motion can be thought of as being produced by one circle rolling on another. The trajectory of a point undergoing one of these motions will be a trochoid.

For the second simple solution we can relax the assumption $r = 0$ in the above, so the equations for θ_1 and θ_2 become non-homogeneous. The right-hand sides are just as in (80) and (81) above but with $\theta_3 = \alpha t + \beta$. The results from (82) and (83) can be used as complimentary functions while we can find the particular integrals using Laplace transforms for example,

$$\theta_1 = -\frac{rt}{\alpha^2}\cos(\alpha t + \beta + \gamma), \tag{84}$$

$$\theta_2 = -\frac{rt}{\alpha^2}\sin(\alpha t + \beta + \gamma). \tag{85}$$

Notice that this motion can be produced by a circle rolling on a uniform spiral.

8 Some Spatial and Rotational Motions

In (Noakes *et al* 1989) the problem of stationary acceleration curves in the rotation group $SO(3)$ was studied. Equations (56) and (57) were derived. In this section we again look for simple solutions.

Inspired by the results for planar motion above we can look for motions generated by rolling one circular cone on another. This will be given by,

$$G(t) = e^{\alpha_1 t\,\mathrm{ad}(\boldsymbol{\omega}_1)}e^{\alpha_2 t\,\mathrm{ad}(\boldsymbol{\omega}_2)}. \tag{86}$$

Notice that here it has been assumed that the variable are linear functions of t, that is α_1 and α_2 are constants. The Lie algebra elements $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are essentially 3-vectors satisfying the standard rules of vector algebra with the commutator $[\mathrm{ad}(\boldsymbol{\omega}_1),\, \mathrm{ad}(\boldsymbol{\omega}_2)]$ represented by the standard vector product, $\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2$. Now with the ansatz (86) above we get,

$$V = \dot{G}G^{-1} = \alpha_1\,\mathrm{ad}(\boldsymbol{\omega}_1) + \alpha_2 e^{\alpha_1 t\,\mathrm{ad}(\boldsymbol{\omega}_1)}\,\mathrm{ad}(\boldsymbol{\omega}_2)e^{-\alpha_1 t\,\mathrm{ad}(\mathbf{s}_1)}, \qquad (87)$$

Hence we have,

$$\begin{aligned} \dot{V} &= \alpha_1\alpha_2 e^{\alpha_1 t\,\mathrm{ad}(\boldsymbol{\omega}_1)}[\mathrm{ad}(\boldsymbol{\omega}_1),\, \mathrm{ad}(\mathbf{s}_2)]e^{-\alpha_1 t\,\mathrm{ad}(\boldsymbol{\omega}_1)}, & (88)\\ \ddot{V} &= \alpha_1^2\alpha_2 e^{\alpha_1 t\,\mathrm{ad}(\boldsymbol{\omega}_1)}[\mathrm{ad}(\boldsymbol{\omega}_1), [\mathrm{ad}(\boldsymbol{\omega}_1), \mathrm{ad}(\boldsymbol{\omega}_2)]]e^{-\alpha_1 t\,\mathrm{ad}(\boldsymbol{\omega}_1)}. & (89) \end{aligned}$$

Then we have,

$$G^{-1}\ddot{V}G = \alpha_1^2\alpha_2 e^{-\alpha_2 t\,\mathrm{ad}(\boldsymbol{\omega}_2)}[\mathrm{ad}(\boldsymbol{\omega}_1), [\mathrm{ad}(\boldsymbol{\omega}_1), \mathrm{ad}(\boldsymbol{\omega}_2)]]e^{\alpha_2 t\,\mathrm{ad}(\boldsymbol{\omega}_2)}. \qquad (90)$$

For this to be constant and hence satisfy equation (59), the vector $[\mathrm{ad}(\boldsymbol{\omega}_1), [\mathrm{ad}(\boldsymbol{\omega}_1), \mathrm{ad}(\boldsymbol{\omega}_2)]]$ must commute with $\mathrm{ad}(\boldsymbol{\omega}_2)$. In familiar 3-vectors this is equivalent to the requirement that $\boldsymbol{\omega}_2$ should be parallel to $\boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2)$. Using the standard expansion for the vector triple product we see that if $\boldsymbol{\omega}_1 \neq \boldsymbol{\omega}_2$, then for solutions we must have $\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2 = 0$. That is the two rotation axes must be perpendicular. Notice that this motion can be produced by bevel gears with perpendicular axes, The ratio α_1/α_2 gives the ratio of the numbers of teeth on the gears.

The argument above applies in any group until we use the vector product. So we can apply this to $SE(3)$ the group of rigid transformations. In this case, the motion will be generated by two screws, $G(t) = e^{t\,\mathrm{ad}(\mathbf{s}_1)}e^{t\,\mathrm{ad}(\mathbf{s}_2)}$. The condition to satisfy is that, $[\mathrm{ad}(\mathbf{s}_1),\, [\mathrm{ad}(\mathbf{s}_1),\, \mathrm{ad}(\mathbf{s}_2)]]$ must commute with $\mathrm{ad}(\mathbf{s}_2)$. In the lie algebra of $SE(3)$ this is equivalent to,

$$< \mathrm{ad}(\mathbf{s}_1),\, \mathrm{ad}(\mathbf{s}_2) >= \mathbf{s}_1^T Q_p \mathbf{s}_2 = 0, \qquad (91)$$

for all p. This implies that the two screws $\mathbf{s}_1$ and $\mathbf{s}_2$ have axes that are perpendicular and meet at a point, the pitches are arbitrary. These paths are difficult to visualise. Figure 2 shows an example of such a path, the perpendicular axes are shown as thick lines,

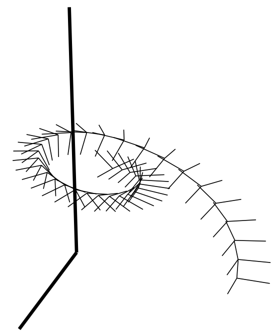

Figure 2: A Stationary Acceleration Curve in Space

the motion is represented by a sequence of orthogonal frames attached to the moving body. In the motion shown the pitch about the horizontal axis is 0 but the motion about the vertical axis has non-zero pitch. As can be seen, the motion can be quite complex.

9 Frenet-Serret Motion

So far we have considered motions determined by exponentials of paths in the Lie algebra or products of such paths. This is not the only way to specify trajectories in the group of rigid body motions. In (Bottema and Roth 1990) Bottema and Roth study a number of 'special motions', one of which is the Frenet-Serret motion. Such a motion is determined by a unit speed space-curve $\mathbf{p}(t)$. Now in a Frenet-Serret motion a point in the moving body moves along the curve and the coordinate frame in the moving body remains aligned with the tangent $\mathbf{t}$, normal $\mathbf{n}$, and binormal $\mathbf{b}$, of the curve. Using the 4-dimensional representation of $SE(3)$ the motion can be specified as,

$$G(t) = \begin{pmatrix} R(t) & \mathbf{p}(t) \\ 0 & 1 \end{pmatrix}, \tag{92}$$

where $\mathbf{p}(t)$ is the curve and the rotation matrix has the unit vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ as columns,

$$R(t) = \big(\mathbf{t} \,\big|\, \mathbf{n} \,\big|\, \mathbf{b}\big). \tag{93}$$

The aim of this section is to determine which of these Frenet-Serret motions can also be stationary acceleration motions. To do this we need to find the velocity of the motion and its derivatives. Of course the famous Frenet-Serret relations will be used to do this:

$$\dot{\mathbf{t}} = \kappa\mathbf{n}, \tag{94}$$

$$\dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b}, \tag{95}$$

$$\dot{\mathbf{b}} = -\tau\mathbf{n}, \tag{96}$$

where κ and τ are respectively the curvature and torsion of the curve. Our work here will be simplified by introducing the Darboux vector $\boldsymbol{\omega} = \tau\mathbf{t} + \kappa\mathbf{b}$ which has the properties that,

$$\dot{\mathbf{t}} = \boldsymbol{\omega} \times \mathbf{t}, \quad \dot{\mathbf{n}} = \boldsymbol{\omega} \times \mathbf{n}, \quad \dot{\mathbf{b}} = \boldsymbol{\omega} \times \mathbf{b},$$

see §10.2 of (Marsh 2005) for example. This means that we can write,

$$\dot{R} = \Omega R, \tag{97}$$

where Ω is the 3×3 anti-symmetric matrix corresponding to $\boldsymbol{\omega}$. Hence we have that,

$$V = \frac{d}{dt}GG^{-1} = \begin{pmatrix} \Omega & \mathbf{t} - \boldsymbol{\omega} \times \mathbf{p} \\ 0 & 0 \end{pmatrix}, \tag{98}$$

remember that $\dot{\mathbf{p}} = \mathbf{t}$ since this is assumed to be a unit speed curve.

Using the Frenet-Serret relations (94)–(96) above the derivative of the velocity is,

$$\dot{V} = \begin{pmatrix} \dot{\Omega} & -\dot{\boldsymbol{\omega}} \times \mathbf{p} \\ 0 & 0 \end{pmatrix}. \tag{99}$$

Here, $\dot{\boldsymbol{\omega}} = \dot{\tau}\mathbf{t} + \dot{\kappa}\mathbf{b}$.

The second derivative of the velocity is now,

$$\ddot{V} = \begin{pmatrix} \ddot{\Omega} & -\ddot{\boldsymbol{\omega}} \times \mathbf{p} - \dot{\kappa}\mathbf{n} \\ 0 & 0 \end{pmatrix}, \tag{100}$$

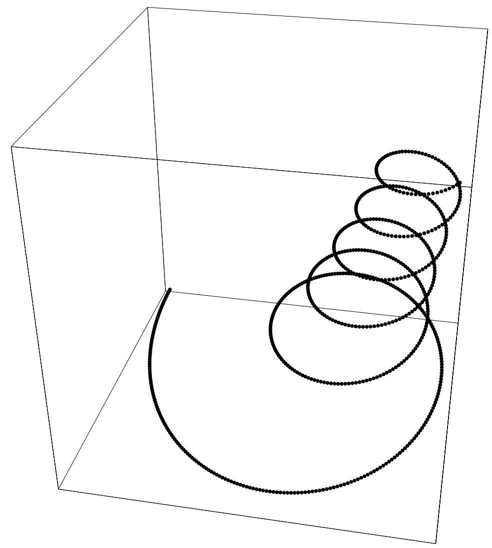

Figure 3: The Curve of a Stationary Acceleration Frenet-Serret Motion

where the fact that $\mathbf{n} = \mathbf{b}\times\mathbf{t}$ has been used here. Also here $\ddot{\boldsymbol{\omega}} = \ddot{\tau}\mathbf{t}+(\dot{\tau}\kappa-\dot{\kappa}\tau)\mathbf{n}+\ddot{\kappa}\mathbf{b}$.

Finally here we compute $G^{-1}\ddot{V}G$, using (92) above this is,

$$G^{-1}\ddot{V}G = \begin{pmatrix} R^T\ddot{\Omega}R & -\dot{\kappa}R^T\mathbf{n} \\ 0 & 0 \end{pmatrix}. \tag{101}$$

Using (93) and the standard formulas for the scalar and vector products of $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ we can expand the above to give,

$$G^{-1}\ddot{V}G = \begin{pmatrix} 0 & -\ddot{\kappa} & \dot{\tau}\kappa-\dot{\kappa}\tau & 0 \\ \ddot{\kappa} & 0 & -\ddot{\tau} & -\dot{\kappa} \\ \dot{\kappa}\tau-\dot{\tau}\kappa & \ddot{\tau} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{102}$$

Setting this equal to a constant gives four differential equations for the two unknowns κ and τ. For solutions to exist a couple of consistency conditions must be satisfied but these are easy to find, if $\dot{\kappa} = u$ a constant then we must have that $\ddot{\kappa} = 0$. Solving for the curvature gives a linear function $\kappa = ut + u_0$. If we substitute κ into the equation $\dot{\tau}\kappa - \dot{\kappa}\tau = w$ the solution for the torsion is another linear function $\tau = \alpha t + \beta$, where the constants satisfy, $\alpha u_0 - \beta u = w$ and of course $\ddot{\tau} = 0$. Standard results curves in

3D tell us that there is a unit-speed curve with these curvature and torsion functions and that, for a given choice of constants, it is unique up to a rigid transformation.

Finding the curve with given curvature and torsion functions involves solving the system of differential equations given by the Frenet-Serret relations (94)–(96) above. This is not straightforward and solutions are only known in a very few cases. However, one classical solutions relevant here is the Cornu spiral. This is the curve that would result if $\alpha = \beta = w = 0$ that is a plane curve. See S 10.6 of (Marsh 2005) for example.

Although there is no classical solution it is always possible to solve the equations numerically. Figure 3 shows a numerically generated curve with curvature $\kappa = t/2+1$ and torsion $\tau = t$.

10 Bishop's Move

In (Bishop 1975) Bishop gives an alternative method to associate a moving frame to points on a curve in 3 dimensions. In the same way that the Frenet-Serret frame determines a special rigid body motion determined by a curve the Bishop frame can also be used to define a special motion. A point in the rigid body follows a curve and an orthonormal frame in the body stays aligned with the Bishop frame. Such a motion will be called a 'Bishop's move' here.

There are some applications of the Bishop frame in Computer graphics to thicken curves and display tubes. The Bishop frame is used because it doesn't 'twist' about the curve. This suggests that the Bishop's moves defined above may be useful for robot path planning.

In this section the velocity of such these Bishop's moves will be computed and stationary acceleration Bishop's moves will be investigated. We begin with the frame equations for the Bishop frame:

$$\dot{\mathbf{t}} = k_1\mathbf{n}_1 + k_2\mathbf{n}_2, \tag{103}$$

$$\dot{\mathbf{n}}_1 = -k_1\mathbf{t}, \tag{104}$$

$$\dot{\mathbf{n}}_2 = -k_2\mathbf{t}. \tag{105}$$

As usual we assume that the curve $\mathbf{p}(t)$ has unit speed and that its tangent vector is given by $\mathbf{t} = \dot{\mathbf{p}}$. The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit normal vectors and together with the tangent vector $\mathbf{t}$, they form an orthonormal frame. So for example $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{t}$ and so forth. The parameters k_1 and k_2 are curvature-like functions. Unlike the Frenet-Serret case, a curve does not uniquely determine a Bishop frame, there is a single rotational freedom in defining the Bishop frame. But if we choose the unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ at $t = 0$ then the Bishop frame for the rest of the curve is unique, (of course the chosen normals must satisfy the frame equations (103)–(105) above).

The path in the group determined by a Bishop's move will be,

$$G(t) = \begin{pmatrix} R(t) & \mathbf{p}(t) \\ 0 & 1 \end{pmatrix}, \tag{106}$$

as before, but now the rotation matrix will be given by,

$$R(t) = \big(\mathbf{t} \,\big|\, \mathbf{n}_1 \,\big|\, \mathbf{n}_2\big). \tag{107}$$

To compute the velocity of a Bishop's move we need an analogue of the Darboux vector. This is given by the vector,

$$\mathbf{a} = -k_2\mathbf{n}_1 + k_1\mathbf{n}_2. \tag{108}$$

It is easy to verify that,

$$\dot{\mathbf{t}} = \mathbf{a} \times \mathbf{t}, \quad \dot{\mathbf{n}}_1 = \mathbf{a} \times \mathbf{n}_1, \quad \text{and} \quad \dot{\mathbf{n}}_2 = \mathbf{a} \times \mathbf{n}_2$$

The velocity is thus,

$$V = \frac{d}{dt}GG^{-1} = \begin{pmatrix} A & \mathbf{t} - \mathbf{a} \times \mathbf{p} \\ 0 & 0 \end{pmatrix}, \tag{109}$$

where, as usual, capital A represents the 3×3 anti-symmetric matrics corresponding to the vector $\mathbf{a}$.

Proceeding as in the previous section we can compute the derivative of the velocity,

$$\dot{V} = \begin{pmatrix} \dot{A} & -\dot{\mathbf{a}} \times \mathbf{p} \\ 0 & 0 \end{pmatrix}, \tag{110}$$

the second derivative of the velocity,

$$\ddot{V} = \begin{pmatrix} \ddot{A} & -\ddot{\mathbf{a}} \times \mathbf{p} - \dot{\mathbf{a}} \times \mathbf{t} \\ 0 & 0 \end{pmatrix}, \tag{111}$$

and finally,

$$G^{-1}\ddot{V}G = \begin{pmatrix} R^T\ddot{A}R & -R^T\dot{\mathbf{a}}\times\mathbf{t} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\ddot{k}_1 & -\ddot{k}_2 & 0 \\ \ddot{k}_1 & 0 & \dot{k}_1k_2 - k_1\dot{k}_2 & -\dot{k}_1 \\ \ddot{k}_2 & k_1\dot{k}_2 - \dot{k}_1k_2 & 0 & -\dot{k}_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{112}$$

Once again setting the above equal to a constant yields four differential equations in the two unknown functions. Again the solution is that both k_1 and k_2 are linear functions of t provided some mild consistency conditions hold. Suppose $k_1 = \alpha_1 t + \alpha_0$ and $k_2 = \beta_1 t + \beta_0$ then if $\dot{k}_1k_2 - k_1\dot{k}_2 = w$ we get the consistency condition $\alpha_0\beta_1 - \alpha_1\beta_0 = w$ and of course $\ddot{k}_1 = \ddot{k}_2 = 0$.

One of the few things known about the functions k_1 and k_2 is that if they lie on a straight line, not containing the origin in k_1-k_2 space, then the corresponding curve lies on a sphere, see (Bishop 1975). Therefore we can see that the curve determining a stationary acceleration Bishop's move lies on a sphere so long as $w \neq 0$.

11 Conclusion

Although it has not been possible to solve the equations for stationary acceleration in general, several special cases have been found. Further, many of these special cases correspond motions that are well know and used in practical situations. Many of these motions can be realised with simple mechanical devices and hence are easily visualised.

Frenet-Serret motions have also been studied, these motions are uniquely determined by a curve in space. It has been shown above that these motions have stationary acceleration if their curvature and torsion functions are linear functions of arc-length. Such curves do seem to have been studied to any great extent.

These ideas led to the definition of a new 'special motion' where the rigid motion follows the Bishop's frame of a curve. These Bishop's moves may be of some interest in robotics since curves in space are well understood and easy to visualise. However, if we impose the extra constraint that the rigid body motion should have stationary acceleration then as we have seen the curve must lie on a sphere.

As a theoretical exercise there is no real reason to prefer motion based on bi-invariant metrics, as here, to those based on left-invariant metrics as in (Žefran and Kumar 1998). Any difference will only be apparent when these ideas are applied to practical problems.

In robotics canonical coordinates of the second kind are familiar from the product of exponentials formula for forward kinematics. Given a particular robot it should be possible to derive the equations for stationary acceleration in $SE(3)$ in terms of the joint angles of the robot and their derivatives. Together with a knowledge of the dynamics of the robot, this might form the basis of a control method to guide the robot along a stationary acceleration path.

In (Žefran *et al* 1998), Žefran et al the jerk of a rigid-body motion is defined as the covariant derivative of the acceleration along the curve. This seems to be by analogy with their definition of the acceleration as the covariant derivative of the velocity along the curve. However, there is another analogy that could be drawn. Along a curve of stationary velocity; a geodesic, the acceleration or $< \dot{V}, \dot{V} >$ is constant. As shown above, along curves of stationary acceleration $< \ddot{V}, \ddot{V} >$ is constant. So perhaps $\ddot{V} = \nabla_V \nabla_V V + (1/2)[\nabla_V V, V]$ would be a good candidate for the jerk of a motion. At least this deserves further investigation.

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